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Large Scale Geometry

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This book is dedicated to our families

Franek, Ania, parents and grandparents;

Alexandria, Andrew, Jenny, parents and grandparents.

Preface

Large scale geometry is the study of geometric objects viewed from afar. In this type of geometry two objects are considered to be the same, if they look roughly the same from a large distance. For example, when viewed from a greater distance the Earth looks like a point, while the real line is not much different from the space of integers. In the past decades mathematicians have discovered many interesting and beautiful large scale geometric properties, which have important applications in topology, analysis, computer science and data analysis.

Historically, large scale geometric ideas appeared in Mostow's rigidity theorem [172], [173] and its generalization due to Margulis [155], [158], as well as in the work of Švarc [227] and later Milnor [166] and Wolf [243] on growth of groups. The impetus came with Gromov's polynomial growth theorem [105] and since then large scale geometric ideas have entered the world of group theory, where they have become a fundamental tool. By now it is well known that many classical notions are in fact of large scale geometric nature, one example being amenability, defined by von Neumann in his work on the Banach–Tarski paradox in 1929.

One of the motivations for this book is the use of large scale geometric methods in index theory. After the Atiyah–Singer index theorem was proved for compact manifolds, a natural question was whether one can extend such index theorems to non-compact manifolds. Connes and Moscovici developed a higher index theory for covering spaces [62]. Motivated by their work, Roe introduced a higher index theory for general non-compact manifolds [207]. Powerful large scale geometric methods have been introduced to compute higher indices for non-compact manifolds [249], [250]. Such computations have allowed us to obtain significant progress on problems like the Novikov conjecture, the Gromov–Lawson–Rosenberg conjecture, or the zero-in-the-spectrum problem, and this area is in a stage of rapid development. More recently, Guentner, Tessera and Yu [117], [118] have introduced large scale geometric methods to study topological rigidity of manifolds. Their method allows them to prove strong results for the stable Borel conjecture and the bounded Borel conjecture.

At the same time, large scale geometric techniques are also of interest in areas such as Banach space geometry, where the coarse classification of Banach spaces remains an interesting problem. Another area of application is theoretical computer science, where embeddability properties, such as compression of coarse embeddings of discrete metric spaces and graphs, allow one to obtain computational efficiency. Finally, we mention that large scale geometric methods have found interesting applications in large data analysis [52], [236].

Our goal is to provide a gentle and fairly detailed introduction to large scale geometric ideas that are used in the study of index problems. We hope that the text will be accessible to a broad audience, in particular to graduate students and newcomers to the field. We provide detailed proofs for most of the theorems stated in the text and every chapter has several exercises at the end. Additionally, some steps in the proofs are left as exercises to the reader. These omissions are always pointed out in the text.

The book is organized in the following way. In the first chapter we discuss basic properties of the coarse category, the geometric viewpoint on finitely generated groups, and we conclude with a section on Gromov hyperbolicity. In the second chapter we give a detailed overview of the notions of asymptotic dimension and decomposition complexity. The third chapter covers amenability of groups. It is slightly shorter, since amenability, as a classical notion, is already a subject of several excellent monographs. In the fourth chapter we discuss the notion of property A for metric spaces and its connections to amenability. In the fifth chapter, coarse embeddings into Banach spaces are studied. The sixth chapter is about affine isometric actions of groups on Banach spaces, in particular a-T-menability (also known as the Haagerup property), Kazhdan's property (T) and constructions of expanders. In the last chapter we introduce elements of large scale algebraic topology: uniformly finite homology and coarse homology theories. The appendix provides a brief survey of applications of coarse geometric properties, discussed in the text, to higher index theory of elliptic operators, and to topological and geometric rigidity.

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Nashville/Warsaw, February 2012 Piotr Nowak, Guoliang Yu

Contents

Notation and conventions

Below we list the notation and conventions used in this book. Most of them are standard.

Sets

$\mathbb{N}$	the set of natural numbers ($0 \notin \mathbb{N}$)
$\mathbb{Z}$	the set of integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$(x_i)_{i \in I}$	a sequence indexed by I
$\coprod_{i \in I} X_i$	for a collection of sets $\{X_i \mid i \in I\}$, the disjoint sum of the sets X_i
$A \triangle B$	symmetric difference of A and B, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$
$\operatorname{supp} f$	the support of a function $f \colon X \to \mathbb{R}$, i.e., the set of those $x \in X$ for which $f(x) \neq 0$
$\operatorname{nbhd}_R(A)$	the R-neighborhood of a subset $A \subseteq X$ of a metric space X, $\operatorname{nbhd}_R(A) = \{x \in X \mid d(x, A) \leq R\}$
$\partial_r A$	the r-boundary of a subset $A \subseteq X$ of a metric space X, defined as $\{x \in X \setminus A \mid d(x, A) \leq r\}$
$\partial^e A$	the edge boundary of a set $A \subseteq V$ in a graph $X = (V, E)$, defined as the set of those edges $e \in E$, which have exactly one vertex in A

Numbers

$\#A$	the number of elements of a set A
$\lvert x \rvert$	the absolute value of $x \in \mathbb{R}$
$\lvert g \rvert$	in a group G, the length of an element $g \in G$
$\lfloor x \rfloor = \sup\{z \in \mathbb{Z} \mid x \geq z\}$	the floor function of $x \in \mathbb{R}$
$(x \mid y)_v$	the Gromov product of x and y with respect to p
$\langle v, w \rangle$	duality pairing of vectors

Groups

$e \in G$	the identity element in the group G
$\mathbb{F}_n$	the free group on n generators
$g \cdot f$	the translation action of $g \in G$ on a function $f \colon G \to X$, defined by $(g \cdot f)(h) = f(hg)$

Spaces

$B_X(x,r)$ or $B(x,r)$	the closed ball of radius r centered at $x \in X$, $B_X(x,R) = \{y \in X \mid d(x,y) \le R\}$
$S_X(x,r)$ or $S(x,R)$	the sphere of radius r centered at $x \in X$, $S_X(x,R) = \{y \in X \mid d(x,y) = R\}$
$f(X)$	the image of the set X under a map f
$\ell_p(I) = \{(x_i)_{i\in I} \mid \sum_{i\in I} \lvert x_i\rvert^p < \infty\}$	the ℓ_p-space on I
$\ell_p(I)_{1,+}$	the set $\{f : I \to \mathbb{R} \mid f(i) \ge 0$ for all $i \in I$ and $\lVert f\rVert_p = 1\}$
$\left(\bigoplus X_n\right)_{(p)}$	the direct sum of Banach spaces X_n with the norm $\lVert (x_n)\rVert_p = (\sum_{i=1}^{\infty} \lVert x_n\rVert^p)^{1/p}$

Functions

1_A	the characteristic function of a set A
1_x	the Dirac function at x

Chapter 1
Metric spaces and large scale geometry

This chapter is devoted to introducing large scale geometry. We start by recalling metric spaces and move on to discuss the natural metric structure of finitely generated groups. Then we describe large scale geometry, first using the notion of quasi-isometries and then in a more general setting, that of coarse maps. The last part of the chapter reviews Gromov hyperbolic spaces.

1.1 Metric spaces

Metric spaces were formally introduced by Fréchet at the beginning of the twentieth century.

Definition 1.1.1. Let X be a set. A *metric* on X is a function $d : X \times X \to [0, \infty)$, such that for all points $x, y, z \in X$ the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*.

The inequality in condition (3) is called the triangle inequality. We will usually simply refer to a metric space as X and omit the direct reference to the metric, unless necessary.

Maps that preserve distances are called isometries.

Definition 1.1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is an *isometry* if

$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x, y \in X$.

Below we list some examples of metric spaces.

Example 1.1.3. Given any set X we can define the discrete metric on X by setting

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. It is straightforward to verify that d is indeed a metric.

Example 1.1.4. Let $X = \mathbb{R}^n$ and let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be points in X. Let

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Then d is the Euclidean metric on $\mathbb{R}^n$. It can also be induced by an inner product on $\mathbb{R}^n$. Set

$$d(x, y) = \sqrt{\langle x - y, x - y \rangle},$$

where $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ is the standard inner product on $\mathbb{R}^n$.

Example 1.1.5. Let $X = \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$. Define

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

This metric on $\mathbb{R}^n$ is often called the *Manhattan metric* since, roughly speaking, this is how we compute distances when walking in a city whose streets follow a grid pattern.

Example 1.1.6. Let $X = \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$. Take the straight line L through x and the origin and define the metric by the formula

$$d(x, y) = \begin{cases} d_E(x, y) & \text{if } y \in L, \\ d_E(x, 0) + d_E(0, y) & \text{otherwise,} \end{cases}$$

where d_E denotes the Euclidean metric on $\mathbb{R}^n$ described in Example 1.1.4. The metric d is called the *train metric* or the *French railway metric*, referencing the situation when various geographical points have train connections to one central station, but not between each other. Traveling between two points that are not collinear with the origin requires taking one train to the central station and another one from the central station to the final destination point.

Note that for the usual Euclidean sphere of radius $1/2$ centered at the origin, viewed as a subspace of the above metric space, this metric realizes the discrete metric described in Example 1.1.3.

Example 1.1.7. Consider a vector space V over $\mathbb{R}$ or $\mathbb{C}$. V is said to be a *normed space* if there is a map $\|\cdot\| \colon V \to [0, \infty)$, satisfying three conditions:

(1) $\|v\| = 0$ if and only if $v = 0$,

(2) $\|\alpha v\| = |\alpha| \, \|v\|$,

(3) $\|v + w\| = \|v\| + \|w\|$,

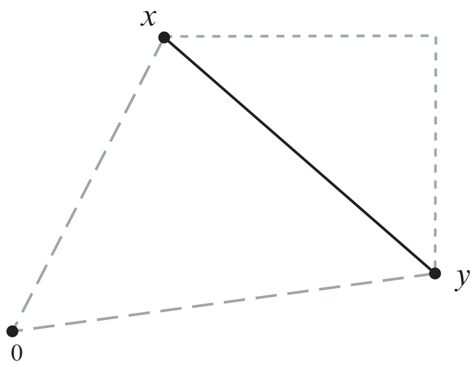

Figure 1.1. Different metrics on the plane. The solid segment realizes the Euclidean metric between x and y. The dashed segment with a right angle realizes the Manhattan metric, and the dashed segment passing through 0 realizes the train metric.

for every $v, w \in V$ and $\alpha \in \mathbb{R}$. $\|\cdot\|$ is called a *norm* on V. Roughly speaking, a norm assigns length to a vector.

A normed space is automatically a metric space. Indeed, a norm on V induces a metric by the formula

$$d(v, w) = \|v - w\|.$$

We leave it as an exercise to check that d is actually a metric.

A normed space is called a *Banach space* if V, with the metric induced by the norm, is a *complete metric space*; that is, every Cauchy sequence converges. The simplest example of a Banach space is the vector space $\mathbb{R}^n$, equipped with the Euclidean norm

$$\|v\| = \Big(\sum_{i=1}^{n} v_i^2\Big)^{1/2},$$

where $v = (v_1, \ldots, v_n)$.

We will now look at examples of Banach spaces. Let $1 \leq p < \infty$. Consider a countable set X and define

$$\ell_p(X) = \big\{f\colon X \to \mathbb{R} \mid \textstyle\sum_{x \in X} |f(x)|^p < \infty\big\}.$$

The space $\ell_p(X)$ is a normed, linear space under coordinate-wise addition and multiplication by scalars, with the norm

$$\|f\|_p = \Big(\sum_{x \in X} |f(x)|^p\Big)^{1/p}.$$

For $p = \infty$ we define

$$\ell_\infty(X) = \{f\colon X \to \mathbb{R} \mid \sup_{x \in X} |f(x)| < \infty\}.$$

This space is also a normed linear space with the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

With the induced metric, the space $\ell_p(X)$ is a Banach space for every $1 \le p \le \infty$. Note that if the set X is infinite, then $\ell_p(X)$ is infinite-dimensional.

We will not verify the fact that the above spaces are Banach spaces but remark that it is not immediate. We refer to [201], [215], [242] for background on ℓ_p-spaces. Note that the spaces $\ell_p(X)$ generalize Examples 1.1.4 and 1.1.5. More precisely, if $X = \{1, 2, \dots, n\}$, then we obtain $\mathbb{R}^n$ with the Euclidean metric for $p = 2$ and with the Manhattan metric for $p = 1$.

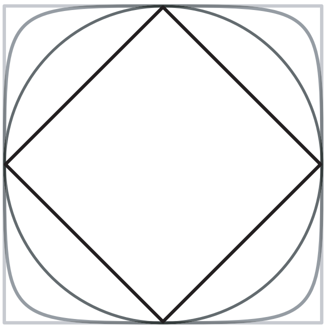

Figure 1.2. Unit spheres in the plane with ℓ_p-metric. The inscribed diamond is the sphere in the Manhattan metric ($p = 1$), the Euclidean ball is round ($p = 2$) and as p grows to infinity the sphere gets closer in shape to the sphere for $p = \infty$, the outer square.

Example 1.1.8. A smooth manifold can be equipped with a Riemannian metric, which turns it into a Riemannian manifold. To define the Riemannian metric recall that a smooth manifold M has a tangent space at every point $x \in M$. Each such tangent space is equipped with an inner product $\langle \cdot\,, \cdot \rangle_x$, which changes smoothly with x. Let $\gamma\colon [0, d] \to M$ be a differentiable curve in M. We define the length of γ by the formula

$$|\gamma| = \int_0^d \|\gamma'(t)\|_x \, dt,$$

where $\|\cdot\|_x$ is the Euclidean norm on the tangent space at $x \in M$. Define the distance between two points $x, y \in M$ by setting

$$d(x, y) = \inf |\gamma|,$$

where the infimum is taken over all curves γ such that $\gamma(0) = x$ and $\gamma(d) = y$.

Example 1.1.9. The hyperbolic space $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid Q(x) = 1,\ x_0 > 0\},$$

where

$$Q(x) = x_0^2 - x_1^2 - \cdots - x_n^2,$$

for $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$. The metric is given by the formula

$$d(x, y) = \operatorname{arccosh}(B(x, y)),$$

where

$$B(x, y) = \frac{(Q(x + y) - Q(x) - Q(y)}{2}.$$

This is the so-called *hyperboloid model of* $\mathbb{H}^n$.

The *Poincaré model*, known also as the disc or the half-plane model, can be described as follows. Consider the unit ball in $\mathbb{R}^n$,

$$\mathbb{B}_n = \{x \in \mathbb{R}^n \mid d(x, 0) < 1\},$$

with distance defined by

$$d(x, y) = \operatorname{arccosh}\left(1 + \frac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}\right),$$

where $\|\cdot\|$ denotes the Euclidean norm. This formula defines a metric and the resulting space is isometric to the hyperbolic space $\mathbb{H}^n$.

Example 1.1.10. Let X be a connected graph; that is X is a pair (V, E), where V is the set of vertices and E is the set of edges. We assume that the edges are not directed. A path of length n between two vertices $x, y \in V$ is a sequence $\{e_i\}_{i=1}^n$ such that x is a vertex of e_1, y is a vertex of e_n and e_i has a common vertex with e_{i+1} for every $i = 1, \ldots, n - 1$.

We can then equip V with a metric by defining the distance $d(x, y)$ between vertices $x, y \in V$ to be the length of the shortest path connecting x and y.

We can also extend this metric to the whole graph by declaring that an edge is isometric to the interval $[0, 1]$, with the usual metric.

The metric above is called a *path metric* on X. It has the useful feature that distances in the graph are realized by paths in the graph. We now formalize this property.

Definition 1.1.11. Let X be a metric space. X is said to be *geodesic* if for any two points $x, y \in X$ there exists an isometric embedding $\gamma\colon [0, d(x, y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x, y)) = y$. The image of γ (which will also be denoted by γ) in X is called a *geodesic between x and y*.

A map $\gamma\colon [0, \infty) \to X$ such that for any $0 \leq a < b$ the restriction $\bar{\gamma}\colon [a, b] \to X$ is a geodesic between $\gamma(a)$ and $\gamma(b)$ is called a *geodesic ray*.

Examples of geodesic spaces are graphs with path metric extended, Euclidean spaces, normed spaces and complete Riemannian manifolds. A discrete metric space is an example of a metric space which is not geodesic. Below we present another example.

Example 1.1.12. The punctured plane $A = \mathbb{R}^2 \setminus \{0\}$, with the metric inherited from Euclidean space, is not geodesic. For example, the distance between the points $(x, 0)$ and $(-x, 0)$, $|x| > 0$, is $2|x|$, but no path between these points in A realizes this distance.

1.2 Groups as metric spaces

Let G be a countable group. We will now introduce a metric on G in such a way that the natural translations on G will be isometries. First we need the notion of a length function.

Definition 1.2.1. Let G be a group. A *length function* on G is a function $|\cdot|\colon G \to [0, \infty)$ such that for any $g, h \in G$ the following conditions are satisfied:

(1) $|g| = 0$ if and only if g is the identity element,
(2) $|g| = |g^{-1}|$,
(3) $|gh| \leq |g| + |h|$.

A reader familiar with normed spaces will quickly realize that a length function on a group bears some resemblance to a norm on a linear space.

We will be interested only in proper length functions. A length function is said to be proper if for any $R > 0$ the set

$$\{g \in G \mid |g| \leq R\},$$

is finite.

Proposition 1.2.2. *On any countable group there exists a proper length function.*

Proof. Choose a symmetric generating set $\Sigma \subseteq G$; that is, for every $\sigma \in \Sigma$ we also have $\sigma^{-1} \in \Sigma$ and every element of G can be written as a finite word in elements of Σ. Let $\beta\colon \Sigma \to [0, \infty)$ be a proper map such that $\beta(\sigma) = \beta(\sigma^{-1})$ and $\beta(e) = 0$ (in case Σ contains the identity element). Index the elements of Σ by the natural numbers so that $\Sigma = \{\sigma_i, \sigma_i^{-1}\}_{i \in \mathbb{N}}$, and define the length function

$$|g|_{\Sigma,\beta} = \min\{\textstyle\sum_{i=1}^{n} \beta(\sigma_i) \mid g = \sigma_1 \ldots \sigma_n \text{ where } \sigma_i \in \Sigma\}.$$

We leave the verification that $|\cdot|_{\Sigma,\beta}$ is indeed a length function as an exercise. The only thing left to prove is that this length function is proper, however this follows easily from the properties of β. $\square$

A countable group G is called *finitely generated* if there exists a finite, symmetric generating set $\Sigma \subseteq G$. In that case every element $g \in G$ can be written as a finite word in elements of Σ,

$$g = \sigma_1 \dots \sigma_i, \tag{1.1}$$

where $\sigma_i \in \Sigma$. In this case we can construct a proper length function by defining the *word length* $|g|_\Sigma$ of $g \in G$ to be the number of generators in the shortest word representing g in (1.1) above:

$$|g|_\Sigma = \min\{n \mid g = \sigma_1 \dots \sigma_n,\ \sigma_i \in \Sigma\}.$$

If we compare this formula with the construction in the proof of Proposition 1.2.2, the above definition corresponds to simply choosing $\beta(i) = 1$ for all $i = 1, \dots, \#\Sigma$. Once a generating set is chosen for a given group we will omit it in the notation, unless necessary.

Example 1.2.3. The length function on $\mathbb{Z}$, associated to the symmetric generating set $\{-1, 1\}$, is simply the absolute value $|z|$ for $z \in \mathbb{Z}$. Similarly, the function $\sum_{i=1}^n |z_i|$, where $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$, is the word length on $\mathbb{Z}^n$, associated to the generating set $\{-1, 1\}^n$.

Example 1.2.4. Let G be a finite group. We can choose the set of generators $\Sigma = G$. The induced word length is characterized by the property $|g| = 1$ for any $g \in G$, $g \neq e$.

There are many possible word lengths on a finitely generated group, and more generally, of proper length functions on a countable discrete group. We leave it to the reader as an exercise to construct different word lengths on various groups.

Because of the similarity between length functions and norms on linear spaces, the following definition is natural.

Definition 1.2.5. Let G be a countable group equipped with a length function $|\cdot|$. The *word length metric* $d = d_{|\cdot|}$, associated to the length function $|\cdot|$, is defined by the formula

$$d(g, h) = |g^{-1}h|$$

for any $g, h \in G$.

We will leave it to the reader as a simple exercise that d is in fact a metric in the sense of Definition 1.1.1. An important property of d is that it is *left-invariant*, i.e., invariant under the left translation action of G on itself.

Indeed,

$$d_G(\gamma g, \gamma h) = |(\gamma g)^{-1}\gamma h| = |g^{-1}h| = d_G(g, h)$$

for any $g, h, \gamma \in G$. Equivalently, one can say that for any element $\gamma \in G$, the left translation by γ, defined to be the map $L_\gamma \colon G \to G$,

$$L_\gamma(g) = \gamma g,$$

is an isometry.

Note that the metric d is proper: a ball of finite radius is a compact (i.e., finite) subset of G. This property is inherited from properness of the length function. In fact, proper length functions on groups correspond exactly to proper, left-invariant metrics: given the latter define $|g| = d(e, g)$ to revert to the length function. To formalize different concepts of properness we introduce the notion of local finiteness and bounded geometry of discrete metric spaces.

Definition 1.2.6. Let X be a metric space. Then

(1) X is called *uniformly discrete* if there exists a constant $C > 0$ such that for any two distinct points $x, y \in X$ we have $d(x, y) \geq C$;

(2) a uniformly discrete metric space X is called *locally finite* if for every $x \in X$ and every $r \geq 0$ we have $\#B(x, r) < \infty$;

(3) a locally finite metric space X is said to have *bounded geometry* if for every $r \in \mathbb{R}$ there exists a number $N(r)$ such that for every $x \in X$ we have $\#B(x, r) \leq N(r)$.

A finitely generated group with a word length metric is a uniformly discrete metric space with bounded geometry.

Except for the few simple examples discussed so far it is usually difficult to visualize a group as a metric space. It is often helpful to fill in the space between the points by joining them by edges, so that the path metric on the graph obtained this way agrees with the metric on the group. This brings us to the concept of the Cayley graph.

Definition 1.2.7. Let G be group generated by a finite, symmetric set $\Sigma \subseteq G$. The *Cayley graph* of G with respect to Σ, denoted by $\mathrm{Cay}(G, \Sigma)$, is defined as follows:

(1) the vertices of $\mathrm{Cay}(G, \Sigma)$ are the elements of G,

(2) vertices $g, h \in G$ are connected by an edge if $g^{-1}h \in \Sigma$.

We equip $\mathrm{Cay}(G, \Sigma)$ with the path metric.

Example 1.2.8. Take $\mathbb{Z}$ with the generating set $\{-1, 1\}$. Then the Cayley graph of $\mathbb{Z}$ is shown in Figure 1.3.

Figure 1.3. Part of the Cayley graph of $\mathbb{Z}$ with the generating set $\{-1, 1\}$.

Example 1.2.9. Let $\mathbb{F}_2$ be the free group generated by $\Sigma = \{a, a^{-1}, b, b^{-1}\}$, subject to no relations.[1] At each vertex there are four edges representing a, b, a^{-1} and b^{-1}.

[1] $\mathbb{F}_2$ is most often referred to as the free group on *two* generators, a and b. Our convention is to require the generating set to be symmetric, thus the natural set of generators of $\mathbb{F}_2$ has four elements.

Since in a free group there are no relations, there are no loops without backtracks (paths with the same beginning and end) in the Cayley graph; see Figure 1.4. Graphs without loops are called *trees*.

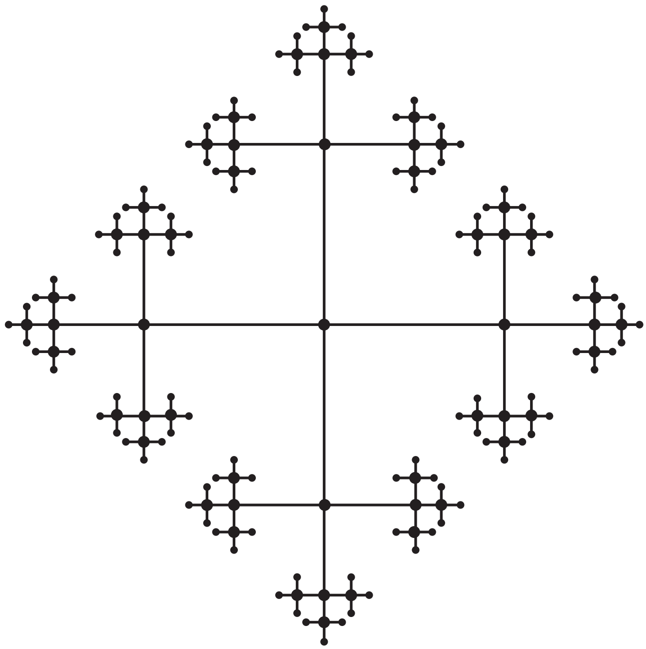

Figure 1.4. Part of the Cayley graph of $\mathbb{F}_2$. Each edge here has length 1.

1.3 Quasi-isometries

The idea behind large scale geometry is to look at spaces only at large scales, to neglect all local, infinitesimal structure (topological or geometric) and consider only those geometric properties that are global. This is described best by a procedure of moving away from a metric space and looking at it from increasing distances. Take, for instance, $\mathbb{Z}$. As we have seen in the previous section, it is a metric space with the natural word length metric associated to the generating set $\{-1, 1\}$ and this metric agrees with the metric inherited from $\mathbb{R}$. Intuitively, $\mathbb{Z}$ should have global geometry similar to the geometry of the real line. However, from the point of view of topology or analysis, which focus on infinitesimal phenomena, $\mathbb{Z}$, as a discrete space, is not interesting at all.

Imagine now that we move away from $\mathbb{Z}$ and start to look at it from a distance. The further away we move, the closer to each other the points of $\mathbb{Z}$ look, as in

Figure 1.5. In the limit, or in other words, from "infinite distance", $\mathbb{Z}$ will resemble the real line. In a similar process a bounded metric space will become a point. This intuition can be generalized using the notion of a net.

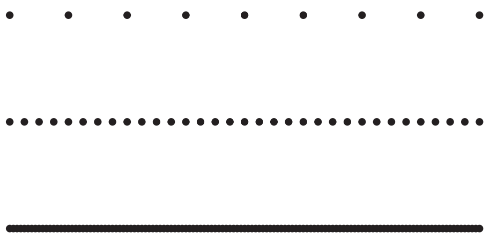

Figure 1.5. Observing $\mathbb{Z}$ from a distance.

Definition 1.3.1. Let X be a metric space and $C > 0$. A subset $N \subseteq X$ is a C-*net* in X if for every $x \in X$ there exists $y \in N$ such that $d_X(x, y) \leq C$. We say that N is a net if it is a C-net for some $C > 0$.

Given a metric space X and $C > 0$ one can always easily construct a C-net $N \subseteq X$ in the following way. Take all C-separated subsets of X, where a set $N \subseteq X$ is C-separated if for any two $x, y \in N$ we have $d(x, y) \geq C$. The collection of all such subsets of X is partially ordered by inclusion. By the Kuratowski–Zorn lemma, there exists a maximal C-separated subset $N \subseteq X$. Then N is a C-net in X. To see this assume the contrary and note that then there would exist a point $x \in X$ for which $d(x, y) \geq C$ for any $y \in N$, which however would contradict maximality of N.

The phenomena described earlier can now be phrased as follows: a metric space X has the same large scale geometry as any net in X. We will now write the above informal discussion in precise, mathematical language.

We start with the notion of a quasi-isometry, and then move on to more general coarse maps and coarse equivalences. In both cases it should be clear that the type of geometry we would like to study requires us to identify maps that look the same from far away. The appropriate notion is the one of close maps.

Definition 1.3.2. Let X and Y be metric spaces and let $f: X \to Y$ and $f': X \to Y$ be maps. We say that f and f' are *close* if there exists a number $C \geq 0$ such that

$$d_Y(f(x), f'(x)) \leq C$$

for all $x \in X$.

Since the distances between the images of f and f' are uniformly bounded over X, close maps indeed look the same from a sufficiently large distance.

Definition 1.3.3. A map $f: X \to Y$ is *large scale Lipschitz* if

(1) there exist constants $L, C > 0$ such that

$$d_Y(f(x), f(y)) \leq L d_X(x, y) + C$$

for any $x, y \in X$, and

(2) f is *metrically proper*; that is, for any bounded set $B \subseteq Y$, the preimage $f^{-1}(B)$ is a bounded subset of X.

The purpose of introducing the additive constant C in the above definition is to control discontinuities: a large scale Lipschitz map can have discontinuities, as long as their size is uniformly controlled. In terms of "looking at spaces from a distance", such discontinuities will eventually disappear if we move far enough. It might be useful to note that for uniformly discrete metric spaces and a large scale Lipschitz map with constants L and C one can always find a greater constant $L' > 0$ such that the map is L'-Lipschitz, i.e., large scale Lipschitz with constants $L = L'$ and $C = 0$.

The following notion defines precisely which objects are the same on the large scale.

Definition 1.3.4. Let X, Y be metric spaces. A map $f: X \to Y$ is called a *quasi-isometry* if the following conditions are satisfied:

(1) there exist constants $L, C > 0$ such that

$$L^{-1} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq L d_X(x, y) + C \tag{1.2}$$

for all $x, y \in X$, and

(2) the image $f(X)$ is a net in Y.

We say that metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f: X \to Y$.

A map which satisfies (1), but not necessarily (2), is called a *quasi-isometric embedding* of X into Y.

A quasi-isometry is called a *bi-Lipschitz map* if the constant C in the above definition is 0. A bi-Lipschitz map is said to be a *bi-Lipschitz equivalence* if it is surjective.

Thus a quasi-isometry looks like a bi-Lipschitz homeomorphism, provided we look at it from sufficiently far away. The above definition does not require a quasi-isometry to be continuous, one-to-one or onto in the usual sense. In fact, quasi-isometries most often do not have these properties. We will illustrate this with several examples.

Example 1.3.5. A surjective isometry $f: X \to Y$ is a quasi-isometry. Let Y_1, Y_2 be bounded metric spaces. Then both the natural projection,

$$X \times Y_1 \xrightarrow{\pi} X,$$

and the natural inclusion,

$$X \xrightarrow{i} Y_2 \times X,$$

are quasi-isometries. The composition

$$X \times Y_1 \xrightarrow{\pi} X \xrightarrow{i} Y_2 \times X$$

is a quasi-isometry as well.

At the beginning of this section we aimed to convince the reader that $\mathbb{Z}$ and $\mathbb{R}$ should, at least intuitively, represent the same object in large scale geometry. The notion of a quasi-isometry allows to make this precise.

Example 1.3.6. Take the "floor map" $\lfloor \cdot \rfloor: \mathbb{R} \to \mathbb{Z}$ which assigns to every real number x the largest integer less or equal than x. The floor map is a quasi-isometry with $L = C = 1$. More generally, let $\mathbb{Z}^n$ be equipped with the word length metric associated to the set of generators $\{-1, 1\}^n$ and let $\mathbb{R}^n$ be equipped with the Manhattan metric (see Example 1.1.5). Then the map

$$\underbrace{\lfloor \cdot \rfloor \times \cdots \times \lfloor \cdot \rfloor}_{n \text{ times}}: \mathbb{R}^n \to \mathbb{Z}^n$$

is a quasi-isometry with constants $L = 1$ and $C = n$. The natural inclusion $\mathbb{Z}^n \subseteq \mathbb{R}^n$ is a quasi-isometry as well.

Example 1.3.7. Let X be a bounded metric space and let $\{p\}$ be a 1-point space. Then the map $\pi: X \to \{p\}$ is a quasi-isometry.

The last two examples are special cases of a more general principle, which is a straightforward consequence of Definition 1.3.4.

Proposition 1.3.8. *Let X be a metric space and N be a net in X. Then the inclusion $i: N \to X$ is a quasi-isometry.*

In particular a group G with a finite generating set Σ and the associated word length metric is quasi-isometric to the Cayley graph $\mathrm{Cay}(G, \Sigma)$ with its path metric. Also, observe that any finite index subgroup H of a group G is a net in G. Thus a group is quasi-isometric to any of its finite index subgroups.

The importance of the above observations lies in the fact that coarse geometry is determined up to nets in metric spaces. For this reason further on in the text we will usually assume that the metric spaces we are dealing with are uniformly discrete.

The next few examples are of maps which are *not* quasi-isometries.

Example 1.3.9. No map from an unbounded space to a bounded space can be a quasi-isometry.

Example 1.3.10. Let $f: \mathbb{R} \to \mathbb{R}^2$ be the inclusion $f(x) = (x, 0)$. Then f is not a quasi-isometry since the image of f is not a C-net in $\mathbb{R}^2$ for any $C \geq 0$. However, f is a quasi-isometric embedding.

Example 1.3.11. Let $\mathbb{N}$ be equipped with the metric inherited from $\mathbb{R}$ and let $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$. Then f is not a quasi-isometry since $d(n, n+1) = 1$, but $d(f(n), f(n+1)) = 2n + 1$, and the latter grows to infinity with n. Thus no constants L and C satisfying the upper bound (1.2) in Definition 1.3.4 exist.

It is clearly a much easier task to argue that a particular map is not a quasi-isometry than to show that two metric spaces are not quasi-isometric. In the next chapters we will introduce several large scale geometric invariants: properties that are preserved by quasi-isometries and allow one to distinguish between metric spaces.

As mentioned in the previous section, the word metric on a group depends on the set of generators. It is natural to ask if large scale geometric properties depend on the set of generators. It turns that this is not the case and all such metrics give rise to the same large scale geometry. This simple, yet fundamental observation is given in the following theorem.

Theorem 1.3.12. *Let G be a group equipped with two finite generating sets Σ, Σ' and associated word length metrics d and d'. Then the metric spaces (G, d) and (G, d') are quasi-isometric.*

Proof. We will show that the identity map $\mathrm{id}_G: (G, d) \to (G, d')$ is a quasi-isometry. Since the metrics are left invariant and id_G is a self-isomorphism of G, observe that to show that id_G is a quasi-isometry it is enough to prove the inequalities

$$|g|' \leq L|g| + C, \tag{1.3}$$

$$|g| \leq L|g|' + C \tag{1.4}$$

for some L and C and all $g \in G$, where $|\cdot|$ and $|\cdot|'$ are length functions associated to sets Σ and Σ', respectively. Let

$$L = \max\{|\sigma'| \mid \sigma' \in \Sigma'\}.$$

For $g \in G$ consider a representation of g as a word $g = \sigma_{i_1}^{\alpha_1} \ldots \sigma_{i_n}^{\alpha_n}$ of length $|g| = \sum \alpha_j$ in Σ. For every $\sigma \in \Sigma$ write σ as a word of length $|\sigma|'$ in Σ'. Then

$$|g|' \leq \sum_{j=1}^{n} \alpha_j |\sigma_{i_j}|' \leq L \sum_{j=1}^{n} \alpha_j = L|g|.$$

This gives the inequality (1.3) with $C = 0$. Interchanging the roles of $|\cdot|$ and $|\cdot|'$ gives the inequality (1.4). □

The above theorem is a fundamental observation in geometric group theory. It says that even though a group G can be equipped with many different proper left-invariant metrics, all these metrics give rise to the same, intrinsic large scale geometry of G. In particular any large scale geometric invariant is an invariant of the group.

In fact, as pointed out in the above proof, $C = 0$ for the quasi-isometry above, which means that word metrics on a finitely generated group are bi-Lipschitz equivalent. We leave it as an exercise to check that a quasi-isometry between two uniformly discrete metric spaces is a bi-Lipschitz equivalence if and only if it is bijective. A natural question is whether quasi-isometry equivalence between two infinite and uniformly discrete metric spaces implies bi-Lipschitz equivalence. The answer turns out to be negative in general [83], but positive under an additional assumption of non-amenability [178], [237]; see also [65], IV.B.46. A related question is if, given a quasi-isometry $f: X \to Y$, there is a bi-Lipschitz map $g: X \to Y$ such that f and g are close. This turns out to be related to uniformly finite homology, discussed in Chapter 7; see Section 7.3.

The next proposition gives one more example of quasi-isometric spaces and has historically been one of the motivating examples in the study of large scale geometry. Recall that an action of a discrete group G on X is properly discontinuous if for every compact set $K \subseteq X$ the set $\{\gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite.

Proposition 1.3.13 (Milnor–Švarc lemma). *Let X be a proper, geodesic metric space and let G be a discrete group acting on X properly discontinuously by isometries. Assume that there is a compact subset of X whose G-translates cover X. Then G is finitely generated and, for a point $x_0 \in X$, the orbit map $f: G \to X$ defined by $f(g) = gx_0$ is a quasi-isometry.*

Proof. Take R sufficiently large so that
$$\bigcup_{g \in G} B(gx_0, R) = X.$$
Let
$$\Sigma = \{g \in G \mid B(gx_0, R) \cap B(x, R) \neq \emptyset\}.$$
Also, define constants
$$L = \inf\{d(B(x_0, R), B(gx_0, R)) \mid g \in G \setminus (\Sigma \cup \{e\})\},$$
and
$$k = \max_{\sigma \in \Sigma} d(x_0, \sigma x_0).$$
Observe that Σ generates G (we leave this as an exercise). Since the translates of $B(x_0, R)$ cover X and G acts on X by isometries, it is enough to show that
$$L|g| - L \leq d(x_0, gx_0) \leq k|g|.$$

The upper bound is clear; we will verify the lower bound. Let m be such that

$$R + (m-1)L \leq d(gx_0, x_0) < R + mL.$$

Choose a sequence of points $x_1, x_2, \ldots, x_{m+1} = gx_0$ satisfying $d(x_0, x_1) < R$ and $d(x_i, x_{i+1}) < L$ for all $i = 1, \ldots, m$. Then

$$|g| \leq m \leq \frac{d(x_0, gx_0)}{L} + \frac{L-R}{L} \leq \frac{1}{L} d(x_0, gx_0) + 1. \qquad \square$$

Let us now revisit Example 1.2.9 of the Cayley graph of $\mathbb{F}_2$. The group $\mathbb{F}_2$ acts on its Cayley graph by isometries, properly discontinuously and with a compact quotient, the figure eight X_E obtained from $[0, 1]$ by identifying the points 0, 1/2 and 1; see Figure 1.6. The fundamental group of X_E is $\pi_1(X_E) = \mathbb{F}_2$. Following this example, a reader familiar with the theory of covering spaces will notice immediately that the Milnor–Švarc lemma implies that the fundamental group of a compact Riemannian manifold M is quasi-isometric to the universal cover of M, with the metric lifted from M.

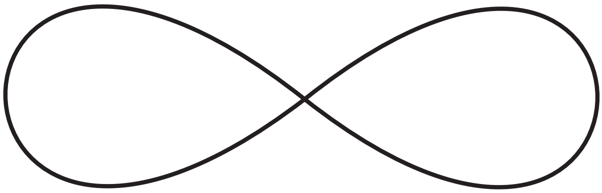

Figure 1.6. The space X_E.

1.4 Coarse equivalences

The notion of a coarse equivalence is more general than that of a quasi-isometry. The difference is that in Definition 1.3.4 we allow more general control functions.

Definition 1.4.1. Let X and Y be metric spaces. A map $f : X \to Y$ is *coarse* if the following two conditions are satisfied:

(1) there exists a function $\rho_+ : [0, \infty) \to [0, \infty)$ such that

$$d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

for all $x, y \in X$, and

(2) f is metrically proper.

Again, just as in the case of quasi-isometries coarse maps are not required to be continuous. Observe also that if X is unbounded, the fact that f is metrically proper implies that ρ_+ cannot be bounded.

Example 1.4.2. A large scale Lipschitz map is a coarse map.

Example 1.4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$. Then f is not a coarse map, since f is not metrically proper.

Definition 1.4.4. Let X and Y be metric spaces. A map $f : X \to Y$ is a *coarse equivalence* if

(1) there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that
$$\lim_{t\to\infty} \rho_-(t) = \infty$$
and the inequality
$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$
holds for all $x, y \in X$, and

(2) the image $f(X)$ is a net in Y.

Two metric spaces X and Y are called *coarsely equivalent* if there exists a coarse equivalence $f : X \to Y$.

A map which satisfies (1), but not necessarily (2), is called a *coarse embedding* of X into Y.

A coarse embedding of X into Y is the same as a coarse equivalence of X with a subspace of Y. Clearly, a quasi-isometry is a coarse equivalence, for which we can choose ρ_- and ρ_+ to be affine functions. However, it is not hard to show that these two notions are different.

Example 1.4.5. Consider the sets $A = \{n^2 \mid n \in \mathbb{N}\}$ and $B = \{n^3 \mid n \in \mathbb{N}\}$, both equipped with the metric inherited from $\mathbb{Z}$. Then the map $n^2 \mapsto n^3$ is a coarse equivalence, but it is not a quasi-isometry. Indeed, we can take $\rho_-(t) = t$ and $\rho_+(t) = t^2$. It is straightforward to check that no affine function can be taken as ρ_+.

Example 1.4.6. Let (X, d) be a metric space. The identity map $I : (X, d) \to (X, \log(d + 1))$ is a coarse equivalence. Indeed, one can take $\rho_-(t) = \rho_+(t) = \log(t + 1)$. However, no affine function can be taken as ρ_-.

Example 1.4.7. Let G be a countable group, which is not necessarily finitely generated. If G is equipped with two proper length functions, then the two metric spaces, obtained by taking G with the induced proper left-invariant metrics, are coarsely equivalent. For instance, consider the infinite direct sum $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ with two different length functions: $|\cdot|_1$, assigning length n to the n-th generator and its inverse, and $|\cdot|_2$, assigning $\sqrt{n}$ to the n-th generator and its inverse. The identity is then a coarse equivalence. Examples as above might not be quasi-isometric.

Example 1.4.8. Let G and H be finitely generated groups. If H is a subgroup of G then the inclusion of H in G is a coarse embedding. Observe that the existence of a coarse embedding of one metric space into another depends only on the quasi-isometry classes of both spaces. Thus we can choose generating sets Σ_H for H and Σ_G for G such that $\Sigma_H \subseteq \Sigma_G$. Then it is easy to verify that the inclusion is a coarse embedding.

In general, the inclusion of a subgroup into the ambient group need not be a quasi-isometric embedding. Consider the discrete Heisenberg group H_3, defined to be the set of upper-triangular matrices

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}.$$

One can then check that the inclusion of the cyclic subgroup generated by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not a quasi-isometric embedding.

Example 1.4.9. Let $F_m(n)\colon \mathbb{N} \to \mathbb{N}$ be defined by the formula

$$F_m(n) = k \quad \text{if } km \le n < (k+1)m.$$

Now define $f\colon \mathbb{N}^2 \to \mathbb{N}^2$,

$$f(m,n) = (m, F_m(n)),$$

for $(m,n) \in \mathbb{N}^2$. Then f is a coarse map but it is not a coarse equivalence onto its image. Indeed, a coarse equivalence $f\colon X \to Y$ between bounded geometry metric spaces must have the property that there is $N > 0$ such that $\# f^{-1}(y) \le N$ for every $y \in Y$.

For a large class of well-behaved metric spaces, such as finitely generated groups, the notions of coarse equivalence and quasi-isometry coincide. This happens if the space satisfies a certain large scale geodesic condition.

Definition 1.4.10. A metric space X is *quasi-geodesic* if there exist constants $L, C > 0$ such that for any points $x, y \in X$ there exists an (L, C)-quasi-isometric embedding $\gamma\colon [0, d(x,y)] \to X$ with $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. The image of γ (also denoted by γ) is called a *quasi-geodesic between x and y.*

In other words, a quasi-geodesic metric space is a space in which the distance between any two points x and y can be, up to a controlled error, realized by a number of steps of length at most $L + C$, where the number of steps depends only on the distance $d(x, y)$. Since the notion of a quasi-geodesic is large scale geometric in nature, we can assume that γ is defined as a map from $[0, d(x, y)] \cap \mathbb{N}$. Similarly as with large scale Lipschitz maps, we can choose the constant L sufficiently large so that $C = 0$.

Example 1.4.11. A space quasi-isometric to a geodesic space is quasi-geodesic.

The following construction will be particularly useful.

Example 1.4.12. Let $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ be a sequence of finite metric spaces. A *coarse disjoint union* is the disjoint union $\coprod X_n$, equipped with the metric d, such that

(1) d restricted to X_n coincides with d_n,

(2) $d(X_i, X_j) \to \infty$ as $i + j \to \infty$ and $i \neq j$.

In other words, the second condition guarantees that the X_i lie far apart from each other in the coarse disjoint union. For instance, the space $C = \{2^n \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$ is a disjoint union of points. A coarse disjoint union is not quasi-geodesic.

The next theorem shows that, for quasi-geodesic spaces, the notions of coarse and large scale Lipschitz maps agree.

Theorem 1.4.13. *Let X be a quasi-geodesic metric space and $f : X \to Y$ be a coarse map. Then f is large scale Lipschitz.*

Proof. Let $x, y \in X$. Since X is quasi-geodesic, there exist constant $L, C > 0$ and a quasi-isometric embedding $\gamma : [0, d(x, y)] \to X$, as in Definition 1.4.10. Let $n = \lceil d(x, y)\rceil$ and $x_i = \gamma(i)$ for $i = 0, 1, \dots, n$, where $x_0 = x$ and $x_n = y$. Then $d(x_i, x_{i+1}) \le L + C$ and

$$\begin{aligned}
d(f(x), f(y)) &\le \sum_{i=0}^{n-1} d(f(x_i), f(x_{i+1})) \\
&\le \rho_+(L + C)\Big(\sum_{i=0}^{n-1} d(x_i, x_{i+1})\Big) \\
&= \rho_+(L + C)\Big(\sum_{i=0}^{n-1} d(\gamma(i), \gamma(i + 1))\Big) \\
&\le \rho_+(L + C)(L + C)n \\
&\le \rho_+(L + C)(L + C)(d(x, y) + 1).
\end{aligned}$$

Thus f is large scale Lipschitz, with both constants equal to $\rho_+(L + C)(L + C)$. □

Corollary 1.4.14. *Let X and Y be quasi-geodesic metric spaces and $f : X \to Y$ be a coarse equivalence. Then f is in fact a quasi-isometry.*

Proof. We apply Theorem 1.4.13 to f and its coarse inverse (see Exercise 1.10). □

The above observation, together with the fact that Cayley graphs of finitely generated groups are quasi-geodesic, shows that to study the large scale geometry of finitely generated groups it suffices to consider quasi-isometries.

Corollary 1.4.15. *Let G and H be finitely generated groups. If $f : G \to H$ is a coarse equivalence, then f is in fact a quasi-isometry.*

1.5 Hyperbolic spaces

The notion of a hyperbolic group was introduced by Gromov [107]. The definition was inspired by a geometric property of triangles in the hyperbolic space $\mathbb{H}^n$.

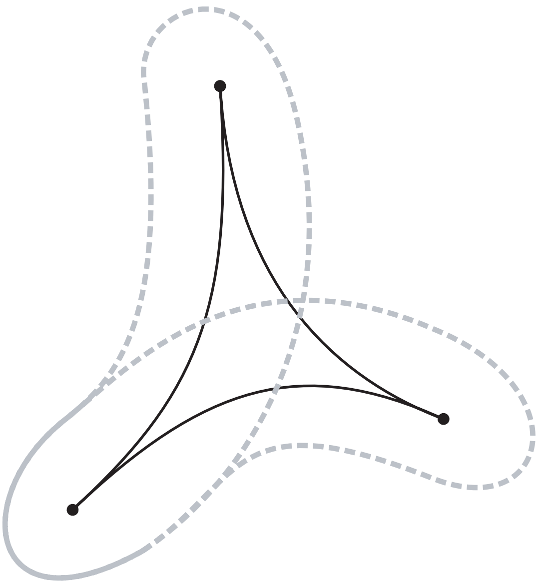

Figure 1.7. A δ-thin triangle. The grey dashed lines show δ-neighborhoods of two of the sides.

A *quasi-geodesic triangle* consists of three vertices x, y, z and quasi-geodesics $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ between the corresponding points. If the quasi-geodesics are geodesic segments, then the triangle is called *geodesic*.

Definition 1.5.1. A geodesic triangle in a metric space is δ-*thin* if the δ-neighborhood of any two sides of the triangle contains the third side.

Given the notion of a δ-thin triangle we can define hyperbolic spaces.

Definition 1.5.2 ([107]). Let $\delta > 0$. A geodesic metric space X is δ-*hyperbolic* if any geodesic triangle in X is δ-thin.

We say that a group G, equipped with a finite generating set Σ, is *hyperbolic* if there exists a $\delta > 0$ such that the Cayley graph $\mathrm{Cay}(G, \Sigma)$ is δ-hyperbolic.

Example 1.5.3. A graph T with the path metric is a metric tree if for any two points there exists exactly one geodesic joining them.

Trees are 0-hyperbolic. Indeed, any geodesic triangle in T is actually a tripod (see Figure 1.8).

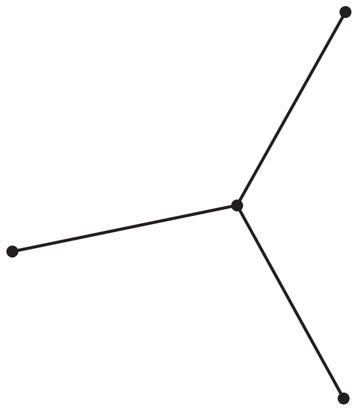

Figure 1.8. A geodesic triangle in a tree is 0-thin.

In particular, the Cayley graph of $\mathbb{F}_2$, with its standard generating set, is a tree shown in Figure 1.4. Thus $\mathbb{F}_2$ is hyperbolic.

In Example 1.1.9 we introduced the hyperbolic space $\mathbb{H}^n$. The following is left as an exercise.

Proposition 1.5.4. *The space $\mathbb{H}^n$ is hyperbolic.*

Example 1.5.5. Let $n \geq 2$. Given a $\delta > 0$ it is not hard to find geodesic triangles in the Euclidean space $\mathbb{R}^n$ that are not δ-thin. Thus the Euclidean space $\mathbb{R}^n$ is not hyperbolic.

Hyperbolicity admits an equivalent definition in terms of the Gromov product.

Definition 1.5.6. Let $x, y, p \in X$. The *Gromov product* of x and y with respect to p is the number

$$(x \mid y)_p = \frac{d(x,p) + d(y,p) - d(x,y)}{2}.$$

Consider a tree T and points $x, y, p \in T$. The geodesic triangle with vertices x, y, p, is a tripod, as in Figure 1.8, but we can view it also as two geodesics σ_x and σ_y, starting at p and ending at x and y, respectively. If we denote by z the point where the two geodesics part, then the Gromov product $(x \mid y)_p$ denotes the length of the geodesic segment from p to z.

Proposition 1.5.7. *A geodesic metric space X is hyperbolic if and only if there exists a number $\delta' \geq 0$ such that*

$$(x \mid y)_p \geq \min\{(x \mid z)_p, (y \mid z)_p\} - \delta'$$

for all $x, y, z, p \in X$.

The following interesting construction indicates that the geodesic condition in the above proposition is essential.

Example 1.5.8. Gromov observed that, given any metric space with a metric d, the new metric $d'(x, y) = \ln(1 + d(x, y))$ always satisfies the inequality in the above proposition. However, this new metric may not be (quasi-)geodesic.

The proof of Proposition 1.5.7 can be found in the literature; see for instance [4] or [37], Ch. III.H. In order to show that hyperbolicity is preserved by quasi-isometries we need to examine the behavior of quasi-geodesics in hyperbolic spaces.

Lemma 1.5.9 (Morse lemma). *Let X be a geodesic, δ-hyperbolic metric space. For any constants $L, C > 0$ there exists a constant $R > 0$ such that if ζ is an (L, C)-quasi-geodesic connecting x and y, then there exists a geodesic γ connecting x and y and contained in the R-neighborhood of ζ.*

Proof. Consider an (L, C)-quasi-geodesic $\zeta\colon [a, b] \to X$. We first construct a continuous, piecewise geodesic path close to ζ. Let $x_1, \ldots, x_n$ be the integer points in $[a, b]$. Let ζ^* be the path consisting of geodesic segments $[\zeta(a), \zeta(x_1)]$, $[\zeta(x_1), \zeta(x_2)], \ldots, [\zeta(x_n), \zeta(b)]$. Clearly, each segment has length at most $L + C$ since ζ and ζ^* coincide at the points $\zeta(a)$, $\zeta(x_i)$ and $\zeta(b)$. We leave it to the reader to verify that ζ^* is an (L, C)-quasi-geodesic, with $C' = 2(L + C)$. It also has the property that

$$\operatorname{length}(\zeta^*[s,t]) \leq K d(\zeta(s), \zeta(t)) + M$$

for $K = L(L + C)$ and $M = (2L + 3)(L + C)$.

Let

$$D = \sup\{d(x, \zeta^*) \mid x \in [\zeta(a), \zeta(b)]\},$$

and let $x_0 \in [\zeta(a), \zeta(b)]$ be such that

$$D = d(x_0, \zeta^*).$$

Let $y_a \in [\zeta^*(a), x_0]$ be the point satisfying $d(y_a, x_0) = 2D$, or let $y_a = \zeta^*(a)$ if $d(x_0, \zeta^*(a)) < 2D$. Define y_b similarly.

There exist points $y_a^*, y_b^* \in \zeta^*$ such that their distances to y_a and y_b, respectively, are at most D. We have

$$d(y_a^*, y_b^*) \leq 6D.$$

Let p be the path connecting y_a, y_a^*, y_b^* and y_b, which runs through ζ^* between y_a^* and y_b^* and is geodesic from y_a to y_a^* and from y_b^* to y_b. We claim that

$$d(x_0, p) \leq \delta |\log_2(\text{length}(p))| + 1.$$

Let $N \geq 0$ be the smallest integer satisfying

$$\text{length}(p) \leq 2^{N+1}.$$

Let $\{P_i\}_{i=0}^{2N}$ be equidistant points on p, with P_0 and P_{2^N} being the endpoints of p.

By hyperbolicity of X, x_0 is within δ of the geodesic $[P_0, P_{2^{N-1}}]$ or within δ of the geodesic $[P_{2^{N-1}}, P_N]$. Iterating this argument we can find a point $Q \in [P_i, P_{i+1}]$ such that $d(x_0, Q) \leq \delta N$. Since

$$d(P_i, P_{i+1}) = \frac{\text{length}(p)}{2^{N+1}},$$

we conclude that

$$d(Q, p) \leq \frac{\text{length}(p)}{2^{N+1}} \leq 1.$$

Consequently,

$$d(x_0, p) \leq \delta N + 1 \leq \delta |\log_2(\text{length}(p))| + 1.$$

At the same time we can estimate the length of p:

$$\begin{aligned} \text{length}(p) &\leq d(y_a, y_a^*) + d(y_b, y_b^*) + K d(y_a^*, y_b^*) + M \\ &\leq 2D + 6DK + M. \end{aligned}$$

Now, since p is at distance greater or equal than D from x_0, we have $d(x_0, p) = D$. Thus we obtain

$$D \leq \delta |\log_2(2D + 6DK + M)| + 1.$$

It follows that there exists a constant $\mathcal{D}$, which depends only on δ, L and C, such that $D \leq \mathcal{D}$. It is easy to check that the geodesic $[\zeta(a), \zeta(b)]$ is in the $\mathcal{D}$-neighborhood of ζ^*.

Now let us consider the $\mathcal{D}$-neighborhood of $[\zeta(a), \zeta(b)]$. We will show that the pieces of ζ^* that lie outside this neighborhood are short. Let $[a', b']$ be the maximal subinterval of $[a, b]$ such that $\zeta^*[a', b']$ lies outside this neighborhood. There exists $z \in [\zeta(a), \zeta(b)]$ such that $d(z, \zeta^*[a, a']) \leq \mathcal{D}$ and $d(z, \zeta^*[b, b']) \leq \mathcal{D}$.

It follows that

$$d(\zeta(a'), \zeta(b')) \leq 2\mathcal{D}$$

and consequently

$$\text{length}(\zeta^*[a', b']) \leq 2\mathcal{D}K + M.$$

Then ζ^* is within $\mathcal{D}(1 + K) + M/2$ of $[\zeta(a), \zeta(b)]$. □

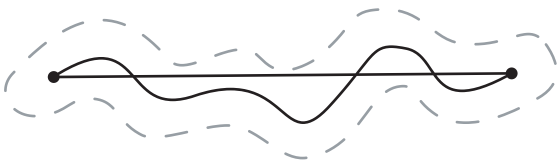

Figure 1.9. Morse lemma. The straight segment is a geodesic and the wavy one is the quasi-geodesic. The grey dashed line encloses the R-neighborhood of the quasi-geodesic.

The large scale invariance of hyperbolicity for geodesic metric spaces is now a straightforward consequence of the Morse lemma.

Theorem 1.5.10. *Let X and Y be quasi-isometric geodesic metric spaces. If Y is hyperbolic then X is hyperbolic.*

Proof. Let $f : X \to Y$ be a quasi-isometry and consider a geodesic triangle τ in X, with vertices x, y, z. The image $f(\tau)$ of this triangle is a quasi-geodesic triangle in Y. By the Morse lemma, there exists $R > 0$ such that the quasi-geodesic segments linking any two of $f(x)$, $f(y)$ and $f(z)$ are contained in R-neighborhoods of the geodesic segments, linking the same vertices. In other words, the quasi-geodesic triangle $f(\tau)$ is contained in an R-neighborhood of a geodesic triangle with vertices $f(x)$, $f(y)$, $f(z)$. Since this geodesic triangle is δ-thin for some $\delta \geq 0$, the quasi-geodesic triangle $f(\tau)$ is δ'-thin, where δ' depends only on δ and R.

Now consider the preimages f^{-1} of two of the δ'-neighborhoods of two sides of the quasi-geodesic triangle $f(\tau)$. Since f is a quasi-isometry, there exists a number $\delta'' > 0$ such that these preimages are contained in the δ''-neighborhoods of the corresponding edges of τ. By construction, the δ''-neighborhoods of these two edges must contain the third edge of τ. Thus τ is δ''-thin and, since the constants were all independent of τ, we see that X is δ''-hyperbolic. □

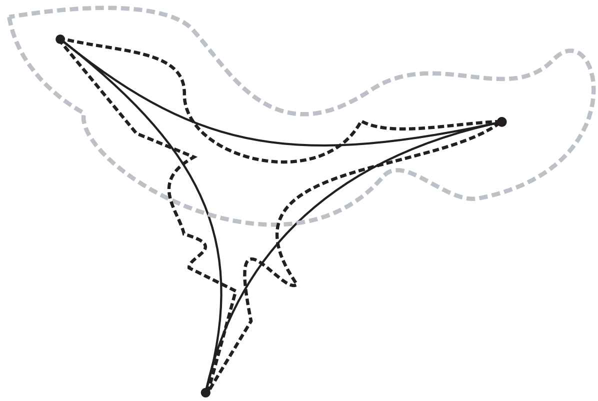

Figure 1.10. Quasi-isometric invariance of hyperbolicity.

Exercises

Exercise 1.1. Verify that the metric in the Poincaré model of $\mathbb{H}^n$, defined in Example 1.1.9, indeed is a metric.

Exercise 1.2. Prove that the 1-dimensional hyperbolic space $\mathbb{H}^1$ and the Euclidean space $\mathbb{R}$ are isometric.

Exercise 1.3. Compute the number of geodesics connecting two points in the Cayley graph of $\mathbb{Z}^n$, with the standard generating set.

Exercise 1.4. Prove that the word length metric (Definition 1.2.5) is a metric.

Exercise 1.5. Draw a part of the Cayley graph of the Baumslag–Solitar group $\mathrm{BS}(1,n)$, which is generated by the elements $\{a, a^{-1}, b, b^{-1}\}$ subject to the relation $ab^n a^{-1} = b$.

Exercise 1.6. Show that a subgroup $H \subseteq G$ is a net in G if and only if H has finite index in G.

Exercise 1.7. Show that the set Σ defined in the proof of the Milnor–Švarc lemma (Proposition 1.3.13) generates the group Γ.

Exercise 1.8. Let T_n denote the infinite tree in which every vertex has degree $n = 2, 3, \ldots$. Let $n, m \geq 2$. Are T_n and T_m quasi-isometric?

Exercise 1.9. Show that a composition of quasi-isometries is again a quasi-isometry. Similarly, prove that a composition of coarse equivalences is again a coarse equivalence.

Exercise 1.10. Prove that $f: X \to Y$ is a quasi-isometry if and only if it is large scale Lipschitz and there exists a large scale Lipschitz map $g: Y \to X$, called the *coarse inverse of* f, such that $f \circ g$ is close to Id_Y and $g \circ f$ is close to Id_X.

Exercise 1.11. Let X, Y be bounded geometry metric spaces and $f: X \to Y$ be a quasi-isometry (respectively, coarse equivalence). Show that there exists a finite metric space Z and a map $f': X \times Z \to Y$ which is also a quasi-isometry (respectively, coarse equivalence) and is onto.

Exercise 1.12. Let X, Y be bounded geometry metric spaces and $f: X \to Y$ be a quasi-isometry (respectively, coarse equivalence). Show that there exists a finite metric space Z and a map $f': X \to Y \times Z$ which is also a quasi-isometry (respectively, coarse equivalence) and is one-to-one.

Exercise 1.13. In Example 1.5.3 it is shown that trees are 0-hyperbolic. Prove the converse, i.e., a 0-hyperbolic graph is a tree.

Notes and remarks

As mentioned in the Preface, large scale geometric ideas appeared as early as Mostow's proof of the rigidity theorem for hyperbolic manifolds [172]. Another remarkable application of large scale geometry was in Gromov's proof of Milnor's conjecture that groups of polynomial growth are exactly the ones that are virtually nilpotent [105]. Gromov implemented the idea of looking at a metric space (X, d) from increasing distances by taking an appropriate limit of the sequence of metric spaces $(X, \frac{1}{n}d)$, as $n \to \infty$. This limit space is known as the asymptotic cone. Later, Gromov's construction was generalized by van den Dries and Wilkie [79].

The notion of coarse embeddings (known also as uniform embeddings) will be studied in detail in Chapter 5.

Roe described an axiomatic way of defining large scale geometric objects without referring to a metric, using a large scale counterpart of uniform structures, called coarse structures. In particular he gave examples of coarse structures which are not induced by a metric; see [209].

Hyperbolic groups were introduced by Gromov [107] and since then have made a huge impact on geometric group theory. There are several references on Gromov's hyperbolicity; see in particular [37], [47], [108], [140], [209].

Chapter 2
Asymptotic dimension and decomposition complexity

Asymptotic dimension is a large scale geometric invariant [108]. It is, roughly speaking, a large scale version of the Lebesgue covering dimension. Decomposition complexity is a far reaching generalization of asymptotic dimension [117], [118]. In this chapter we introduce these notions and study their properties.

2.1 Topological dimension

We will start by briefly reviewing topological dimension for separable metric spaces. Dimension is one of the most natural and intuitive geometric invariants. To understand the necessity of this notion consider the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$. Both are linear spaces, and it is easy to show that they are not linearly isomorphic – as vector spaces their dimension is m and n, respectively. Since this number does not change under linear isomorphisms, we obtain our claim.

However if we forget about the linear structure and view $\mathbb{R}^m$ and $\mathbb{R}^n$ only as topological spaces, the question of how to decide whether the two are homeomorphic does not have an immediate answer. The notion of topological dimension was introduced exactly to deal with such questions. The first, most natural idea is to define dimension inductively as follows:

(1) $\dim X = -1$ if and only if $X = \emptyset$;
(2) $\dim X \leq n$ if for every point $x \in X$ and every open neighborhood $V \subseteq X$ of x there exists an open set $U \subseteq V$ such that $x \in U$, $\bar{U} \subseteq V$ and $\dim \partial U \leq n-1$;
(3) $\dim X \geq n$ if it is not true that $\dim X \leq n-1$.

The reader can easily check that according to this definition a point has dimension 0 and that $\dim \mathbb{R}^n = n$. Inductive dimension is preserved by homeomorphisms, and the dimension of a subspace is not greater than the dimension of the ambient space. This approach to dimension is very intuitive, however, in many cases a different definition is more convenient. Recall that the multiplicity of a cover is the largest integer n such that every point $x \in X$ is contained in at most n elements of the cover. Given a cover $\mathcal{U}$, a cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$.

Definition 2.1.1. Let X be a separable, metric space. The *covering dimension* (or simply *topological dimension*) of X is the smallest number $n \in \mathbb{N} \cup \{-1\}$ such that every open cover has a refinement of multiplicity at most $n+1$.

The above definition can be applied to any topological space. Moreover, for well-behaved spaces, such as separable metric spaces, the covering dimension and the inductive dimension agree.

Example 2.1.2. Any interval $[a,b] \subseteq \mathbb{R}$ satisfies $\dim[a,b] \leq 1$. Indeed, take any open cover $\mathcal{U} = \{U_i\}$ of $[a,b]$ and recall that any open set on the real line is a disjoint union of open intervals. Thus each U_i is such a disjoint union of open intervals. We can refine $\mathcal{U}$ by a cover consisting of open intervals, which in turn has a refinement such that each point is contained in at most two intervals.

Example 2.1.3. More generally, $\dim \mathbb{R}^n = n$. The fact that $\dim \mathbb{R}^n \leq n$ can be easily verified. The inequality $\dim \mathbb{R}^n \geq n$ is harder to prove using only the definition of covering dimension, and we will assume it without proof. We will similarly assume that $\dim[0,1]^n = n$. We refer to [84] for a complete argument.

One of the main features of topological dimension is that it is a topological invariant.

Theorem 2.1.4. *Let X and Y be separable metric spaces and assume that X and Y are homeomorphic. Then*

$$\dim X = \dim Y.$$

This gives an answer to our initial question about homeomorphic Euclidean spaces: $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $m = n$.

2.2 Asymptotic dimension

Asymptotic dimension is a large scale geometric version of the covering dimension and to introduce it we need to "coarsen" the notion of multiplicity. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of a metric space. Given $R > 0$, the R-multiplicity of $\mathcal{U}$ is the smallest integer n such that for every $x \in X$ the ball $B(x,R)$ intersects at most n elements of $\mathcal{U}$.

Definition 2.2.1. Suppose that X is a metric space. The *asymptotic dimension* of X is the smallest number $n \in \mathbb{N} \cup \{0\}$ such that for every $R > 0$ there exists a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i\in I}$ with R-multiplicity $n+1$. It is denoted by $\operatorname{asdim} X = n$.

If we are not interested in the exact value of $\operatorname{asdim} X$, but we know that there exists an $n \in \mathbb{N}$ such that $\operatorname{asdim} X \leq n$, then we say that X has finite asymptotic dimension and write $\operatorname{asdim} X < \infty$.

Example 2.2.2. Consider the set $\{n^2 \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$ with the metric induced by the absolute value. This metric space has asymptotic dimension 0.

Example 2.2.3. We will show that asdim $\mathbb{Z} \leq 1$. For every $R > 0$ we need to exhibit a cover of $\mathbb{Z}$ satisfying Definition 2.2.1. Given $R > 0$, consider the family of sets $\mathcal{U} = \{\{2nR, \dots, 2(n+1)R - 1\}\}_{n \in \mathbb{N}}$. Since these sets are disjoint and each one of them is of length $2R$, a ball of radius R centered at any point $x \in \mathbb{Z}$ intersects at most two elements of $\mathcal{U}$. Similarly one can show that asdim $\mathbb{Z}^n \leq n$. This is left as an exercise, see also Figure 2.1.

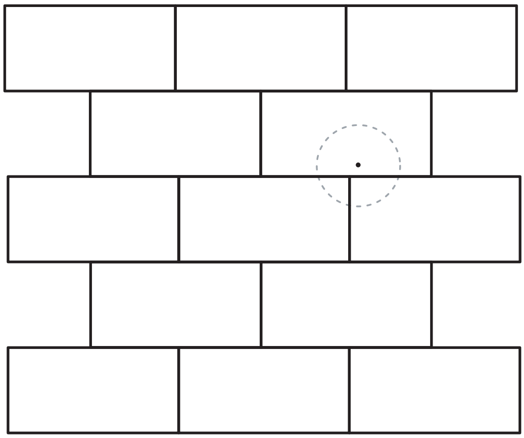

Figure 2.1. The above tiling, taken at larger and larger scales, shows that asdim $\mathbb{Z}^2 \leq 2$.

It might be surprising that it is quite hard to prove that asdim $\mathbb{Z}^n$ is *equal* to n without appealing either to the topological dimension theory or to the methods of coarse algebraic topology. We will show this fact in Example 2.2.6, under the assumption that the topological dimension of $\mathbb{R}^n$ is n. We will also discuss an argument that uses coarse homology in Proposition 7.6.5.

The first property of asymptotic dimension is monotonicity under passing to subspaces.

Proposition 2.2.4. *Let X be a metric space and let $A \subseteq X$ be a subspace. Then*

$$\operatorname{asdim} A \leq \operatorname{asdim} X.$$

Proof. Let $R > 0$ and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X with R-multiplicity asdim $X + 1$. Consider the cover $\mathcal{V} = \{A \cap U_i\}_{i \in I}$. Since for any $x \in A$ we have $B_A(x, R) = B_X(x, R) \cap A$, the R-multiplicity of $\mathcal{V}$ is at most the multiplicity of $\mathcal{U}$. □

Asymptotic dimension is also a large scale geometric invariant.

Theorem 2.2.5. *Let X and Y be coarsely equivalent metric spaces. Then*

$$\operatorname{asdim} X = \operatorname{asdim} Y.$$

Proof. We will prove this fact for quasi-isometric spaces, the general case of coarse invariance is left as an exercise. Let $f : X \to Y$ be a quasi-isometry with constants $L, C > 0$. Let $r > 0$ and $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of Y satisfying Definition 2.2.1 with $R = Lr + C$. Consider the cover of X given by

$$\mathcal{V} = \{f^{-1}(U_i)\}_{i \in I}.$$

This cover is uniformly bounded and we need to show that its r-multiplicity is bounded by asdim $X + 1$.

By construction, $B_X(x, r)$ intersects as many elements of $\mathcal{V}$ as the number of elements of $\mathcal{U}$ intersecting $f(B_X(x, r))$. But for every $x \in X$ and a ball $B_X(x, r)$ the image satisfies $f(B_X(x, r)) \subseteq B_Y(f(x), Lr + C) = B_Y(f(x), R)$ by the choice of R. Since $B_Y(y, R)$ intersects at most asdim $Y + 1$ elements of $\mathcal{U}$, we obtain the inequality

$$\operatorname{asdim} X \leq \operatorname{asdim} Y.$$

The same argument applied to the coarse inverse of f (which also can be assumed to be a quasi-isometry) proves the opposite inequality. $\square$

We will now use the above theorem and prove that asdim $\mathbb{Z}^n > n - 1$ under the assumption that $\dim[0, 1]^n = n$.

Example 2.2.6. Let $n \in \mathbb{N}$ and assume that asdim $\mathbb{Z}^n \leq n - 1$. Then for every $R > 0$ there exists a uniformly bounded cover $\mathcal{U}$ of $\mathbb{Z}^n$ with R-multiplicity n. The map $\lfloor \cdot \rfloor^n : \mathbb{R}^n \to \mathbb{Z}^n$ (product of n floor maps) is a quasi-isometry and we consider the cover $\mathcal{V}$, consisting of the preimages of elements of $\mathcal{U}$, as in the proof above. Consider the cover $\widetilde{\mathcal{V}} = \{\mathrm{nbhd}_\varepsilon(V) \mid V \in \mathcal{V}\}$. This cover is an open cover of $\mathbb{R}^n$ of multiplicity n.

Consider now the cube $I^n = [0, 1]^n \subseteq \mathbb{R}^n$ and let $\mathcal{U}$ be an open cover of I^n. We will use the cover $\widetilde{\mathcal{V}}$ to construct a refinement of $\mathcal{U}$ of multiplicity n and this will lead us to a contradiction. Since I^n is a compact metric space, there exists a Lebesgue number for the cover $\mathcal{U}$, i.e., a $\delta > 0$ such that for every set $A \subseteq I^n$ satisfying diam $A \leq \delta$ there exists $U \in \mathcal{U}$ with $A \subseteq U$. For the cover $\widetilde{\mathcal{V}}$ we consider its image $\mathcal{R}$ under the dilation $x \mapsto \alpha x$ of $\mathbb{R}^n$, with α chosen in such a way that $\alpha D \leq \delta$, where D is the uniform upper bound on the diameter of elements of $\widetilde{\mathcal{V}}$. Since the diameter of any element of $\mathcal{R}$ is at most δ, the restriction of $\mathcal{R}$ to I^n is a refinement of $\mathcal{U}$. Moreover, the dilation does not change multiplicity, so the refinement has multiplicity n. This shows that any open cover of I^n has a refinement of multiplicity n and implies that $\dim I^n \leq n - 1$, which is a contradiction.

Combining Theorem 2.2.5 with the above fact we have the following

Corollary 2.2.7. *The groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are coarsely equivalent if and only if $n = m$.*

Another invariant that can be used to distinguish between $\mathbb{Z}^n$ and $\mathbb{Z}^m$ up to quasi-isometries is volume growth, which will be discussed in Chapter 3.

2.3 Dimension of hyperbolic groups

This section is devoted to proving a result due to Gromov [108], that hyperbolic groups have finite asymptotic dimension. To get the proper intuition for what to expect in the general case we will first examine the case of a tree.

Proposition 2.3.1. *Let T be a tree. Then* $\operatorname{asdim} T \leq 1$.

Proof. Given a tree T, fix a vertex $v \in T$, let $R > 0$ and for $n = 0, 1, 2, \dots$ construct the annuli

$$A_n^R = \{x \in T \mid Rn \leq d(x, v) \leq R(n+1)\}.$$

The annuli A_n^R satisfy the R-multiplicity condition, however the cover by A_n^R's is not uniformly bounded and we further subdivide each annulus A_n to overcome this. The subdivision will be given as equivalence classes $A_{n,i}^R$ $(i = 1, 2, \dots)$ of the relation on A_n^R defined by saying that $x \sim y$ if $(x \mid y)_v \geq R(2n-1)$. The cover $A_{n,i}^R$ has all the required properties. It is uniformly bounded since if $x, y \in A_{n,i}^R$, then

$$d(x, y) = d(x, v) + d(y, v) - 2(x \mid y)_v \leq 4R(n+1) - 2R(2n-1) = 6R.$$

The $(R/2)$-multiplicity is still 2 since the $A_{n,i}^R$ are at least distance $R/2$ apart. Indeed, if x and y are in different sets we have

$$(x \mid y)_v = d(x, v) + d(y, v) - 2d(x, y) \leq 2R(n-1),$$

which implies the estimate. □

Since the Cayley graph of a free group is a tree, we conclude that the asymptotic dimension of a free group does not exceed 1. On the other hand, since a free group contains an isometric copy of the infinite cyclic group $\mathbb{Z}$, we have $\operatorname{asdim} \mathbb{F}_n \geq \operatorname{asdim} \mathbb{Z} = 1$ by Proposition 2.2.4.

Corollary 2.3.2. *The asymptotic dimension of the free group $\mathbb{F}_n$ equals* 1.

An interesting consequence of the above is that surjective homomorphisms do not preserve asymptotic dimension. Indeed, any finitely generated group, in particular $\mathbb{Z}^n$, is a quotient of a free group, but, as we have seen, $\operatorname{asdim} \mathbb{Z}^n = n$. Note also that in the above proof we have not used any assumptions on degrees of vertices of T. This means that the above theorem applies to any tree, even one with vertices of infinite degree.

The next statement is Gromov's theorem on finiteness of asymptotic dimension of hyperbolic groups. The proof, taken from [210], closely resembles the case of a tree.

Theorem 2.3.3. *Let G be a hyperbolic group. Then* $\operatorname{asdim} G < \infty$.

Proof. For a given $R > 0$ consider the annuli

$$A_n^R = \{g \in G \mid nR \le |g| \le (n+1)R\}.$$

Let $S(e,k) = \{g \in G \mid |g| = k\}$ denote the sphere in G of radius k. For each $n \in \mathbb{N}$ choose a maximal R-separated subset $\{m_{n,i}\} \subseteq S(e, nR)$ and write

$$A_{n,i}^R = \big\{g \in A_n^R \mid (g \,|\, m_{n,i})_e \ge (n - \tfrac{1}{2})R - \delta\big\}.$$

First we need to show that $A_{n,i}^R$ is indeed a cover of G. When x belongs to the annulus A_n^R, we denote by $p(x)$ the point where the geodesic from e to x intersects the sphere $S(e, nR)$. Then, clearly,

$$|p(x)| = nR = (x \,|\, p(x))_e.$$

By the choice of the $m_{n,i}$ there exists an index i such that $d(m_{n,i}, p(x)) \le R$. This implies that

$$(p(x) \,|\, m_{n,i})_e \ge \left(n - \frac{1}{2}\right) R.$$

Since G is δ-hyperbolic we have

$$(x \,|\, m_{n,i})_e \ge \min\{(x \,|\, p(x))_e,\ (p(x) \,|\, m_{n,i})_e\} - \delta \ge \left(n - \frac{1}{2}\right) R - \delta.$$

This however means that $x \in A_{n,i}^R$, and we have indeed constructed a cover of G. Note that this cover is uniformly bounded since for any $x \in A_{n,i}^R$ we have

$$d(m_{n,i}, x) = |m_{n,i}| + |x| - 2(m_{n,i} \,|\, x)_e \le 2R + 2\delta.$$

It remains to show that the R-multiplicity of the cover is finite. Assume that $y \in A_n$ and that $A_{n,i}$ intersects the ball $B(y, R)$. Let $x \in A_{n,i} \cap B(y, R)$ and let $q(y)$ be the intersection of the geodesic from e to y with the sphere of radius $(n - \frac{1}{2})R$. Clearly,

$$|q(y)| = \left(n - \frac{1}{2}\right) R = (y \,|\, q(y))_e.$$

Note also that

$$(y \,|\, x)_e \ge \left(k - \frac{1}{2}\right) R$$

and that

$$(m_{n,i} \,|\, x)_e \ge \left(n - \frac{1}{2}\right) R - \delta$$

by previous calculations. Then

$$(q(y) \mid x)_e \geq \min\{(q(y) \mid y)_e, (x \mid y)_e\} - \delta \geq \left(n - \frac{1}{2}\right) - 2\delta$$

and

$$(m_{n,i} \mid q(y))_e \geq \min\{(m_{n,i} \mid x)_e, (q(y) \mid y)_e\} - \delta \geq \left(n - \frac{1}{2}\right) - 3\delta.$$

Thus

$$d(m_{n,i}, q(y)) = |m_{n,i}| + |q(y)| - 2(m_{n,i} \mid q(y))_e \leq \frac{1}{2}R + 6\delta.$$

Thus the number of elements of the cover that intersect the ball $B(x, R)$ is at most the number of elements of the ball of radius $\frac{1}{2}R + 6\delta$, which is bounded. □

It is worth noting that even though hyperbolic groups "look like trees from far away", their asymptotic dimension may attain arbitrarily large values. Indeed, one can prove that the n-dimensional hyperbolic space $\mathbb{H}^n$ satisfies $\operatorname{asdim} \mathbb{H}^n = n$. We let G be a discrete group of isometries of $\mathbb{H}^n$ acting properly discontinuously with a compact quotient. By the Milnor–Švarc lemma, G is quasi-isometric to $\mathbb{H}^n$ and

(1) G is hyperbolic by Theorem 1.5.10,

(2) $\operatorname{asdim} G = \operatorname{asdim} \mathbb{H}^n$ by Theorem 2.2.5.

2.4 Upper bounds for asymptotic dimension

In this section we present a product-type upper bound for asymptotic dimension, proved by Bell and Dranishnikov [28]. We begin with a simple example.

Example 2.4.1. Let X and Y be metric spaces of finite asymptotic dimension. The Cartesian product can be equipped with a metric by the formula

$$d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

Then $X \times Y$ also has finite asymptotic dimension. Let $\mathcal{U}$ and $\mathcal{V}$ be uniformly bounded covers of R-multiplicity $\operatorname{asdim} X + 1$ and $\operatorname{asdim} Y + 1$, respectively. Define the cover $\mathcal{W}$ of $X \times Y$ by

$$\mathcal{W} = \{U \times V \mid U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}.$$

This cover is uniformly bounded and we will show that it has finite R-multiplicity. Indeed, since

$$B_{X \times Y}((x, y), R) \subseteq B_X(x, R) \times B_Y(y, R),$$

and the latter set intersects at most $(\operatorname{asdim} X + 1)(\operatorname{asdim} Y + 1)$ elements of the cover $\mathcal{W}$, so does $B_{X \times Y}((x, y), R)$. We thus obtain the estimate

$$\operatorname{asdim} X \times Y \leq (\operatorname{asdim} X + 1)(\operatorname{asdim} Y + 1) - 1.$$

We want to view this example as a special case of a more general construction. Let G and H be finitely generated groups and write $X = G$, $Y = H$. Consider the action of the group $G \times H$ on H defined by $(g,h)h' = hh'$ for $g \in G$, $h, h' \in H$. Then $G \times \{e_H\}$ acts trivially on H and the above example can be interpreted as follows: asdim *of the group acting is at most* asdim $+1$ *of the space upon which the group acts, times* asdim $+1$ *of the* (*largest*) *subgroup which acts trivially*.

Definition 2.4.2. Let G be a finitely generated group acting on a metric space by isometries and let $R > 0$. The *R-stabilizer* $\operatorname{Stab}_R(x)$ of the point $x \in X$ is the set

$$\operatorname{Stab}_R(x) = \{g \in G \mid d(x, gx) \leq R\}.$$

For $R = 0$ the R-stabilizer reduces to the standard notion of a stabilizer of a point and the coarse stabilizer is supposed to "look like a stabilizer from a distance".

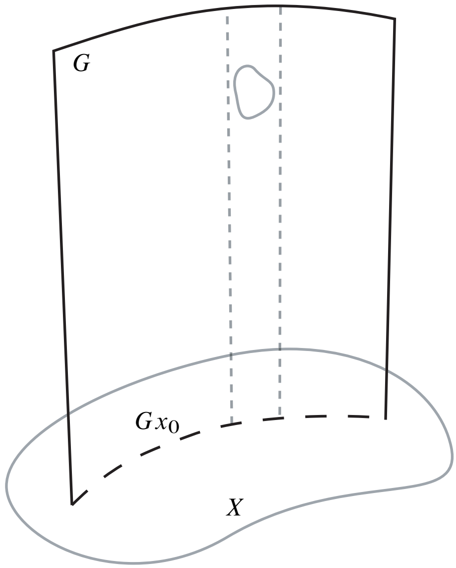

Figure 2.2. Fibering theorem for asymptotic dimension. The horizontal dashed line is the orbit of x_0 under the action of G, while the vertical dashed lines depict $\pi^{-1}(U)$.

Theorem 2.4.3 ([28]). *Let X be a metric space and $x_0 \in X$. Let G act on X by isometries and let $\kappa > 0$ be such that* $\operatorname{asdim} \operatorname{Stab}_R(x_0) \leq \kappa$ *for every $R > 0$. Then*

$$\operatorname{asdim} G \leq (\operatorname{asdim} X + 1)(\kappa + 1) - 1.$$

Proof. Choose a finite symmetric generating set Σ of G and consider the map $\pi : G \to X$ sending g to its image in the orbit of x_0,

$$\pi(g) = gx_0.$$

Then
$$\mathrm{Stab}_R(x_0) = \pi^{-1}(B_X(x_0, R)).$$
Define
$$\lambda = \max\{d(x_0, \sigma x_0) + 1 \mid \sigma \in \Sigma\}$$
(the extra 1 is to guarantee that $\lambda \geq 1$). Since G acts by isometries, we have
$$d_X(\pi(g), \pi(g\sigma)) = d_X(gx_0, g\sigma x_0) = d_X(x_0, \sigma x_0).$$
It follows that π is λ-Lipschitz.

Let $R > 0$. There exists a uniformly bounded cover $\mathcal{U}$ of the orbit Gx_0 of x_0 with R-multiplicity $\operatorname{asdim} X + 1$. Set $S = \sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}\}$. There exists a uniformly bounded cover $\mathcal{V}$ of $\mathrm{Stab}_S(x_0)$ with R-multiplicity at most $\kappa + 1$. For every element $U \in \mathcal{U}$ we choose one element $g_U \in G$ such that $g_U x_0 \in U$. We define a collection $\mathcal{W}$ of subsets of G by setting
$$\mathcal{W} = \{g_U V \cap \pi^{-1}(U) \mid U \in \mathcal{U},\ V \in \mathcal{V}\}.$$

We need to check all the conditions from the definition of asymptotic dimension. The family $\mathcal{W}$ is a cover of G. To see this let $g \in G$ and let $gx_0 \in U$. We have $g_U^{-1}U \subseteq B(x_0, S)$ since $\operatorname{diam} U \leq S$. Thus $g_U^{-1}gx_0 \in B(x_0, S)$, which means exactly that $g_U^{-1}g \in \mathrm{Stab}_S(x_0)$. This on the other hand means that $g_U^{-1}g$ belongs to some set $V \in \mathcal{V}$ and, consequently, $g \in g_U V$.

The cover $\mathcal{W}$ is uniformly bounded. Indeed, if W is an element of $\mathcal{W}$, then $\operatorname{diam} W \leq \operatorname{diam} g_U V \leq \operatorname{diam} V$ for some $V \in \mathcal{V}$ and $U \in \mathcal{U}$, and the claim follows from the fact that the cover $\mathcal{V}$ is uniformly bounded.

Finally, we need to show that the R-multiplicity of the cover $\mathcal{W}$ is bounded by $(\operatorname{asdim} X + 1)(\kappa + 1)$. Since π is λ-Lipschitz and the R/λ-multiplicity of $\mathcal{U}$ is at most $\operatorname{asdim} X + 1$ (note that $\lambda \geq 1$), we have that a ball of radius R in G can intersect at most $\operatorname{asdim} X + 1$ sets of the form $\pi^{-1}(U)$. On the other hand, since the R-multiplicity of $\mathcal{V}$ is bounded by $\kappa + 1$ and multiplication by g_U is an isometry, we see that the ball of radius R can intersect at most $\kappa + 1$ sets of the form $g_U V$ inside each $\pi^{-1}(U)$. All this gives that the R-multiplicity of the cover $\mathcal{W}$ is bounded above by $(\operatorname{asdim} X + 1)(\kappa + 1)$, as required. □

We will now apply the above theorem to show that certain classes of groups have finite asymptotic dimension.

Recall that given groups N, G, Q we say that G is an extension of N by Q if $N \subseteq G$ is a normal subgroup and $G/N = Q$. A special case of an extension is the semi-direct product. By $\mathrm{Aut}(G)$ we denote the group of self-isomorphisms of G.

Definition 2.4.4. Let G and H be groups and let $\theta\colon H \to \mathrm{Aut}(G)$, $h \mapsto \theta_h$, be a homomorphism. The *semi-direct product* $G \rtimes_\theta H$ of G and H relative to θ is the group whose elements are elements of $G \times H$ with the group operation
$$(g_1, h_1)(g_2, h_2) = (g_1 \theta_{h_1}(g_2), h_1 h_2).$$

The simplest case of the semi-direct product is the direct product. This corresponds to θ being the trivial homomorphism. If θ is not trivial, then it introduces "twisting" in the first coordinate, while the second coordinate behaves as in the direct product. The semi-direct product is a special case of a group extension since the subgroup $G \subseteq G \rtimes_\theta H$ is normal and $(G \rtimes H)/G$ is isomorphic to H. These facts are left as exercises.

We first introduce a metric on the quotient group that ensures that the quotient map is 1-Lipschitz. Let Σ be a finite, symmetric generating set for G. Denote the quotient map by $q\colon G \to G/N$. Then the set $q(\Sigma) \subseteq G/N$ is a generating set of the quotient group, and we equip G/N with the word length metric associated to $q(\Sigma)$. The group G acts on the quotient group G/N by left multiplication in the usual way: $g[h] = [gh]$ where $g \in G$ and $[h] \in G/N$ is the coset represented by $h \in G$.

Lemma 2.4.5. *Let G be a finitely generated group and N a normal, finitely generated subgroup. Then for the action of G on G/N we have* $\mathrm{Stab}_R(e) = \{x \in G \mid d(x,N) \le R\}$ *for every $R > 0$.*

Proof. Let $g \in \mathrm{Stab}_R(e) \subseteq G$. Then $|q(g)| \le R$. This means that there exists a sequence $\sigma_i \in \Sigma$, $i = 1,\dots,k$ with $k \le R$, such that $q(g) = q(\sigma_1)q(\sigma_2)\dots q(\sigma_k)$. Let $h = \sigma_1\sigma_2\dots\sigma_k$. We then have

$$d(g, gh^{-1}) \le |h^{-1}| \le R,$$

but this already means that $d(g,N) \le R$ since $gh^{-1} \in N$, and we obtain the inclusion $\mathrm{Stab}_R(e) \subseteq \{x \in G \mid d(x,N) \le R\}$.

Conversely, if $d(g,N) \le R$, then $d(g,h) \le R$ for some $h \in N$. Thus $d(q(g),e) \le R$, which gives $\mathrm{Stab}_R(e) \supseteq \{x \in G \mid d(x,N) \le R\}$. ☐

The next theorem gives an upper bound on the asymptotic dimension of an extension.

Theorem 2.4.6. *Let G be a finitely generated group and N a normal, finitely generated subgroup of G. Then*

$$\operatorname{asdim} G \le (\operatorname{asdim} N + 1)(\operatorname{asdim} G/N + 1) - 1.$$

Proof. By the previous lemma and Theorem 1.3.12, $\mathrm{Stab}_R(e)$ is coarsely equivalent to N for every $R > 0$. Thus, by Theorem 2.2.5,

$$\operatorname{asdim} \mathrm{Stab}_R(e) = \operatorname{asdim} N$$

for every $R > 0$, and we apply Theorem 2.4.3 to prove the claim. ☐

2.5 Asymptotic dimension of solvable groups

Recall that a topological group is a group with a topology compatible with its group operations. The following is a basic notion of group theory.

Definition 2.5.1. A topological group G is *solvable* if G has a sequence of closed subgroups

$$\{e\} = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = G \tag{2.1}$$

such that N_{i-1} is normal in N_i for every $i = 2, 3, \ldots, k$ and the factor groups N_i / N_{i-1} are abelian.

In other words, a solvable group is a group that can be obtained by taking an abelian group, extending it by another abelian group and repeating the process of extending by an abelian group finitely many times.

Example 2.5.2. The group Sol,[1] defined as $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where the $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

is solvable. Indeed, we have the following sequence of subgroups

$$\{e\} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2 \subseteq \mathrm{Sol},$$

where $\mathrm{Sol} / \mathbb{Z}^2$ is isomorphic to $\mathbb{Z}$.

Corollary 2.5.3. *If G is a finitely generated solvable group and G has a sequence of subgroups N_i as in* (2.1) *such that each factor group N_i / N_{i-1} is finitely generated for every i, then* $\operatorname{asdim} G < \infty$.

The proof is an application of Theorem 2.4.3 and we leave it to the reader. We remark that the same strategy does not work without the additional assumption of finite generation of the quotient groups. As we will see soon, in general abelian groups might not have finite asymptotic dimension.

If G is a Lie group, we choose a left G-invariant metric on G. Such a metric can be constructed by using a left G-invariant Riemannian metric on G. Left G-invariant metrics on G are coarsely equivalent to each other.

Definition 2.5.4. A Lie group G is said to be *almost connected* if it has finitely many connected components.

[1] The group Sol defined here is a lattice in the real Lie group **Sol**, giving rise to one of 3-dimensional geometries appearing in the geometrization conjecture. Any choice of an integer matrix A with $\mathrm{trace}(A) > 2$ also gives such a lattice.

An almost connected Lie group is coarsely equivalent to a solvable Lie group. Indeed, for an almost connected Lie group G there is an exact sequence

$$\{e\} \to G_e \to G \to K \to \{e\},$$

where e is the identity of G, G_e is the connected component of e and K is finite. It follows that G is quasi-isometric to G_e. Consequently,

$$\operatorname{asdim} G \le \operatorname{asdim} G_e,$$

and we need to show that the latter is finite. For a connected Lie group H, such as G_e, we can obtain an exact sequence of Lie groups

$$\{e\} \to A \to H \to \mathrm{GL}(n, \mathbb{R}) \to \{e\},$$

where A is an abelian Lie group, using the adjoint representation of H on the Lie algebra, i.e., the tangent space of H at the identity. More precisely, if an element h in the connected Lie group H is in the kernel of the adjoint representation (i.e., the representation $\eta \to g^{-1}\eta g$ for all η in the tangent space of H at the identity), then h commutes with elements of the type $\exp(\eta)$. However, under the assumption that H is connected, H is generated by elements of the type $\exp(\eta)$, i.e., each group element is a product of elements of the type $\exp(\eta)$. This implies that h is in the center of H. Hence the kernel of the adjoint representation is abelian. The group A, as a locally compact abelian Lie group, has finite asymptotic dimension.

Now, for the linear group $\mathrm{GL}(n, \mathbb{R})$, using the Gram–Schmidt orthogonalization process one obtains the QR-decomposition of any matrix into a product of a lower triangular matrix and an orthogonal matrix:

$$\mathrm{GL}(n, \mathbb{R}) = \mathrm{LT}(n, \mathbb{R})\, \mathrm{O}(n, \mathbb{R}).$$

Here $\mathrm{LT}(n, \mathbb{R})$ is the Lie group of all lower triangular invertible matrices and $\mathrm{O}(n, \mathbb{R})$ is the Lie group of all orthogonal matrices. It is not difficult to verify by induction on n that $\mathrm{LT}(n, \mathbb{R})$ is solvable; we leave this fact as an exercise. Hence, $\mathrm{LT}(n, \mathbb{R})$ has finite asymptotic dimension, by the following lemma.

Lemma 2.5.5. *If G is an almost connected solvable Lie group, then* $\operatorname{asdim} G < \infty$.

The above lemma follows from a Lie group analogue of Theorem 2.4.3 (see Exercise 2.4) and the fact that any sequence of closed subgroups N_i in (2.1) of an almost connected Lie group has the property $\operatorname{asdim}(N_i/N_{i-1}) < \infty$. The same analogue of Theorem 2.4.3, together with Corollary 2.5.5, gives

Theorem 2.5.6. *Let G be an almost connected Lie group. Then G has finite asymptotic dimension.*

We can also deduce the following

Corollary 2.5.7. *If* Γ *is a discrete subgroup of an almost connected Lie group, then* $\operatorname{asdim}\Gamma < \infty$.

The above corollary implies that the discrete group $\mathrm{SL}(n, \mathbb{Z})$, consisting of $n \times n$ matrices with integer entries and determinant 1, has finite asymptotic dimension.

2.6 Groups with infinite asymptotic dimension

Having seen various examples of groups with finite asymptotic dimension we might ask whether there exist groups which do not have finite asymptotic dimension. The answer is affirmative. We start with an example of an infinitely generated group with such a property.

Example 2.6.1. Let $\mathbb{Z}^{(\infty)} = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ be equipped with the proper, left invariant metric

$$d_{(\infty)}(x, y) = \sum_{i=1}^{\infty} i\,|x_i - y_i|,$$

where $x = (x_1, x_2, \ldots, x_i, \ldots)$, $y = (y_1, y_2, \ldots, y_i, \ldots)$ are points in $\mathbb{Z}^{(\infty)}$. Then

$$\operatorname{asdim} \mathbb{Z}^{(\infty)} = \infty.$$

Indeed, note that for every $n \in \mathbb{N}$ there is an inclusion

$$j_n \colon \mathbb{Z}^n \to \mathbb{Z}^{(\infty)},$$

mapping $\mathbb{Z}^n$ onto the first n copies of $\mathbb{Z}$ in $\mathbb{Z}^{(\infty)}$. Moreover, j_n is a quasi-isometry between $\mathbb{Z}^n$ with the standard word length metric and its image in $\mathbb{Z}^{(\infty)}$. Since asymptotic dimension is inherited by subspaces (Proposition 2.2.4) and is invariant under quasi-isometries (Theorem 2.2.5), we see that

$$\operatorname{asdim} \mathbb{Z}^{(\infty)} \geq \operatorname{asdim} \mathbb{Z}^n = n.$$

This estimate holds for every $n \in \mathbb{N}$, and it follows that $\operatorname{asdim} \mathbb{Z}^{(\infty)}$ cannot be finite.

To construct finitely generated groups with infinite asymptotic dimension, we introduce the notion of a wreath product. Let G and H be finitely generated groups. Consider the set $C_c(H; G)$ of finitely supported functions $f \colon H \to G$. This set has a natural group structure through coordinate-wise multiplication and as a group can be identified with $(\bigoplus_{h \in H} G)$. Note that the action of H on itself by translations induces an action of H on $C_c(H; G)$, and we write $(\theta_h(f))(h') = f(h^{-1}h')$ for $h, h' \in H$ and $f \in C_c(H; G)$.

Definition 2.6.2. The *wreath product* $G \wr H$ of G and H is defined to be the semi-direct product

$$C_c(H; G) \rtimes_\theta H.$$

If G and H are generated by finite sets Σ_G and Σ_H respectively, then the wreath product $G \wr H$ is generated by $\Sigma_H \times \{1\} \cup \{1\} \times \Sigma_H$ (Exercise 2.5), and consequently is a finitely generated group.

Proposition 2.6.3. *The group $\mathbb{Z} \wr \mathbb{Z}$ has infinite asymptotic dimension.*

Proof. It suffices to show that the group $\mathbb{Z} \wr \mathbb{Z}$ contains a subgroup isomorphic to $\mathbb{Z}^n$ for every $n \in \mathbb{N}$. Indeed, since asymptotic dimension is monotone under passing to subspaces (Proposition 2.2.4) and is coarsely invariant (Theorem 2.2.5), inclusion of a finitely generated subgroup is a coarse embedding (Example 1.4.8), we see that

$$\operatorname{asdim} \mathbb{Z} \wr \mathbb{Z} \geq \operatorname{asdim} \mathbb{Z}^n = n$$

for every $n \in \mathbb{N}$, similarly as in the previous example. The elements

$$\{\gamma^k \sigma \gamma^{-k} \mid k = 1, 2, \ldots, n\},$$

where γ and σ are the obvious images under inclusion of the generators of the two copies of $\mathbb{Z}$, generate the required subgroup isomorphic to $\mathbb{Z}^n$. □

2.7 Decomposition complexity

In this section we introduce a far-reaching generalization of asymptotic dimension, called decomposition complexity. It was introduced in [117] and studied in [118]. For $R > 0$ we say that a family $\mathcal{U}$ of subsets of a metric space X is *R-disjoint* if

$$d(U, U') \geq R$$

for every distinct pair $U, U' \in \mathcal{U}$. We will also say that the family is *S-bounded*, where $S > 0$, provided that $\operatorname{diam} U \leq S$ for every $S \in \mathcal{U}$. Similarly, a family is said to be bounded if it is S-bounded for some S.

To motivate the definition of finite decomposition complexity, we first state the following characterization of asymptotic dimension. The proof is left to the reader (see Exercise 2.3).

Theorem 2.7.1. *Let X be a metric space. The following conditions are equivalent:*

(1) $\operatorname{asdim} X \leq n$,

(2) *for every $R > 0$ there exist $n+1$ families $\mathcal{U}_i$ of subsets of X, $i = 0, 1, \ldots, n$, and $S > 0$ such that each $\mathcal{U}_i$ is R-disjoint, S-bounded and the families $\mathcal{U}_i$ cover X.*

The above characterization is at the origin of finite decomposition complexity. By a metric family we mean a family of metric spaces.

Definition 2.7.2. Let $R > 0$. A family of metric spaces $\mathcal{X}$ is *R-decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition

$$X = X_0 \cup X_1,$$

such that

$$X_i = \coprod X_i^j, \quad i = 0, 1,$$

where $X_i^j \in \mathcal{Y}$ and each $\{X_i^j\}_{j \in \mathbb{N}}$ is an R-disjoint family for $i = 0, 1$.

We will write $\mathcal{X} \xrightarrow{R} \mathcal{Y}$ to denote that $\mathcal{X}$ R-decomposes over $\mathcal{Y}$. Consider the following "decomposition game".

The decomposition game. The game has two players, A and B and a metric family $\mathcal{Y}_0$ as a starting point.

Round 1: Player A claims that he or she can decompose the family $\mathcal{Y}_0$. B challenges A by giving an integer R_0. The first round ends when player A R_0-decomposes the family $\mathcal{Y}_0$ over some family $\mathcal{Y}_1$.

Round k: Player A now claims that he or she can decompose the family $\mathcal{Y}_{k-1}$. Similarly, B gives $R_{k-1} \in \mathbb{N}$ and the round ends when A finds an R_{k-1}-decomposition of $\mathcal{Y}_{k-1}$ over some family $\mathcal{Y}_k$.

Player A wins if for some $k \in \mathbb{N}$ he or she can decompose $\mathcal{Y}_k$ over a bounded family. We say that player A has a winning strategy if, regardless of the choices of R_i made by the player B, player A can always win the game.

Definition 2.7.3. A metric family $\mathcal{X}$ has *finite decomposition complexity* if player A has a winning strategy in the decomposition game with $\mathcal{Y}_0 = \mathcal{X}$.

A metric space X has finite decomposition complexity if the family $\{X\}$ has finite decomposition complexity.

Since finite decomposition complexity was motivated by finite asymptotic dimension, it is natural to ask about the relation between them.

Proposition 2.7.4. *Let X be a metric space satisfying* $\operatorname{asdim} X \leq 1$. *Then X has finite decomposition complexity.*

Proof. Consider the decomposition game with $\mathcal{Y}_0 = \{X\}$. Player B gives $R = R_0$. Player A decomposes X, using the definition of asymptotic dimension, into two families of subsets of X, which are all uniformly bounded in diameter and each family is R-disjoint. Thus A wins the game every time, regardless of the choice of R. $\square$

Note that a winning game for player A consists of a sequence of decompositions

$$\mathcal{X} = \mathcal{Y}_0 \xrightarrow{R_0} \mathcal{Y}_1 \xrightarrow{R_1} \mathcal{Y}_2 \xrightarrow{R_3} \cdots \xrightarrow{R_n} \mathcal{Y}_n, \tag{2.2}$$

in which $\mathcal{Y}_n$ is a bounded family. Describing a winning strategy is thus equivalent to finding a sequence similar to (2.2) for every possible choice of R_i. This can be described rigorously.

Definition 2.7.5. A *decomposition tree* is a directed rooted tree such that every vertex other than the root has a unique incoming edge, either zero or countably infinitely many outgoing edges, labeled by natural numbers and every oriented geodesic starting at the root is finite.

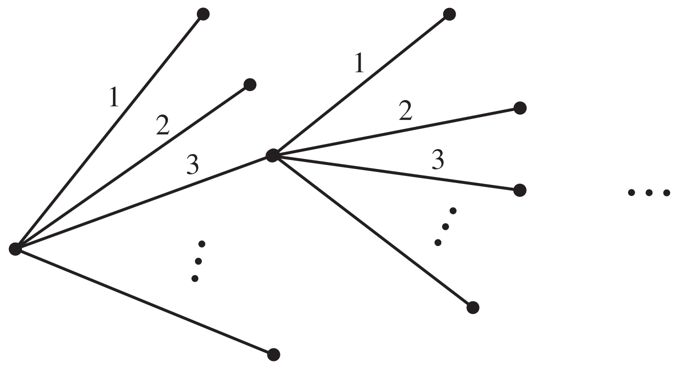

Figure 2.3. A decomposition tree.

Let $\mathcal{X}$ be a family of metric spaces. A winning decomposition strategy for $\mathcal{X}$ is a decomposition tree T in which every vertex is labeled by a family $\mathcal{Y}$ of metric spaces in such a way that

(1) the root is labeled by $\mathcal{X}$,

(2) given an edge labeled by a natural number R, if the initial vertex is labeled by $\mathcal{Y}$ and the terminal vertex is labeled by $\mathcal{Y}'$ then $\mathcal{Y}$ R-decomposes over $\mathcal{Y}'$,

(3) each vertex with no outgoing edges is labeled by a bounded family.

We will now discuss briefly notions similar to finite decomposition complexity. One can define a weak version of finite decomposition complexity as follows.

Definition 2.7.6. Let $R > 0$. A family of metric spaces $\mathcal{X}$ is $(R; d)$-*decomposable* over a metric family $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits a decomposition

$$X = X_0 \cup \cdots \cup X_d,$$

such that

$$X_i = \coprod X_i^j, \quad i = 0, 1, \ldots, d,$$

where $X_i^j \in \mathcal{Y}$ and each $\{X_i^j\}_{j \in \mathbb{N}}$ is an R-disjoint family for $i = 0, 1, \ldots, d$.

A metric family $\mathcal{X}$ decomposes weakly over the collection $\mathcal{A}$ of metric families if there exists $d \in \mathbb{N}$ such that for every $R > 0$ there exist $Y \in \mathcal{A}$ and an $(R; d)$-decomposition of $\mathcal{X}$ over $\mathcal{Y}$.

Thus R-decomposability used earlier is the same as $(R, 1)$-decomposability in the above sense. The class of metric families with weak decomposition complexity is the class containing bounded families and closed under weak decomposition.

At this point we would like to recall the notion of asymptotic property C, defined by Dranishnikov [68]. This notion can be viewed as a large scale analogue of the topological property C of Haver.

Definition 2.7.7. A metric space X has *asymptotic property C* if for every sequence of numbers $R_1 \le R_2 \le R_3 \le \cdots$, there is a finite sequence of uniformly bounded families of open subsets $\{\mathcal{U}_i\}_{i=1}^n$ such that the union $\bigcup_i = \mathcal{U}_i$ is a covering of X and every family $\mathcal{U}_i$ is R_i-disjoint.

Clearly, finite decomposition complexity implies the weak finite decomposition complexity, and weak finite decomposition complexity implies asymptotic property C, but it is not clear whether the converses hold.

Open question 2.7.8. *Does weak finite decomposition complexity imply finite decomposition complexity?*

Open question 2.7.9. *Does asymptotic property C imply weak finite decomposition complexity?*

2.8 Invariance and permanence

To prove large scale invariance of finite decomposition complexity we need a notion of a coarse map that would be appropriate for dealing with families of spaces. We introduce such maps below. The notion of a decomposition strategy allows us then to give unified, simple proofs of coarse invariance and a fibering theorem for spaces with finite decomposition complexity.

Definition 2.8.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be families of metric spaces.

(1) A collection of maps $F = \{f : X \to Y\}_{X \in \mathcal{X}}$, where $Y \in \mathcal{Y}$, is a *map of families*, denoted by $F : \mathcal{X} \to \mathcal{Y}$.

(2) A map of families $F : \mathcal{X} \to \mathcal{Y}$ is *uniformly expansive* if, for every $X \in \mathcal{X}$, there exists a nondecreasing function $\rho_+ : [0, \infty) \to [0, \infty)$ such that

$$d(f(x_1), f(x_2)) \le \rho_+(d(x_1, x_2))$$

for every $f \in F$ and all $x_1, x_2 \in X$.

(3) A map of families $F\colon \mathcal{X} \to \mathcal{Y}$ is *effectively proper* if, for every $X \in \mathcal{X}$, there exists a proper nondecreasing function $\rho_-\colon [0,\infty) \to [0,\infty)$ such that

$$\rho_-(d(x_1,x_2)) \le d(f(x_1), f(x_2))$$

for every $f \in F$ and all $x_1, x_2 \in X$.

(4) A map of families $F\colon \mathcal{X} \to \mathcal{Y}$ is a *coarse embedding* if it is effectively proper and uniformly expansive.

Given a metric family $\mathcal{X}$, a subspace of $\mathcal{X}$ is a family $\mathcal{Y}$ such that every $Y \in \mathcal{Y}$ is a subspace of some $X \in \mathcal{X}$.

Lemma 2.8.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be families of metric spaces and $F\colon \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map of families. Then for every $R > 0$ there exists $S > 0$ such that if $\mathcal{V}$ and $\mathcal{W}$ are subspaces of $\mathcal{Y}$ and $\mathcal{V}$ S-decomposes over $\mathcal{W}$, then the inverse image $F^{-1}(\mathcal{V})$ R-decomposes over $F^{-1}(\mathcal{W})$.*

Proof. Given F let ρ_+ be as in Definition 2.8.1. We will show that $S = \rho_+(R)$ satisfies the required conditions. Let $\mathcal{V}$ R-decompose over $\mathcal{W}$. Then any $V \in \mathcal{V}$ admits a decomposition

$$V = V_0 \cup V_1,$$

where

$$V_i = \coprod_{S\text{-disjoint}} V_i^j$$

and $V_i^j \in \mathcal{W}$ for every i, j. For any $V \in \mathcal{V}$ take the inverse image $f^{-1}(V)$ for some $f \in F$. Then this inverse image admits a decomposition

$$f^{-1}(V) = f^{-1}(V_0) \cup f^{-1}(V_1),$$

where

$$f^{-1}(V_i) = \bigcup f^{-1}(V_i^j). \tag{2.3}$$

By the definition of S one can easily see that the disjoint sum (2.3) is R-disjoint. $\square$

Lemma 2.8.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be families of metric spaces and let $F\colon \mathcal{X} \to \mathcal{Y}$ be an effectively proper map of families. For any bounded subfamily $\mathcal{V} \subseteq \mathcal{Y}$ the inverse image $F^{-1}(\mathcal{V})$ is a bounded subfamily of $\mathcal{X}$.*

Proof. Let F be as above and let ρ_- be the function from Definition 2.8.1 corresponding to the effective properness of F. Let $C > 0$ be such that

$$\operatorname{diam} V \le C$$

for every $V \in \mathcal{V}$. Then, for $\rho_-(D) \geq C$, the family $F^{-1}(\mathcal{V})$ satisfies

$$\operatorname{diam} F^{-1}(V) \leq D$$

for every V. □

Theorem 2.8.4. *Let $\mathcal{X}$, $\mathcal{Y}$ be families of metric spaces. If $\mathcal{X}$ coarsely embeds into $\mathcal{Y}$ and $\mathcal{Y}$ has finite decomposition complexity, then $\mathcal{X}$ has finite decomposition complexity.*

Proof. Let $F : \mathcal{X} \to \mathcal{Y}$ be a coarse embedding. Consider the decomposition strategy for $\mathcal{Y}$, with a decomposition tree T. Let s_i be an increasing sequence of natural numbers satisfying

$$s_i \geq i.$$

We now modify and relabel the tree T as follows. For every edge labeled by a number $a \notin \{s_1, s_2, \dots\}$ we remove this edge together with the whole tree rooted at the end of this edge.

We now relabel each edge of the form

$$\mathcal{V} \xrightarrow{s_i} \mathcal{W}$$

in the remaining tree by

$$F^{-1}(\mathcal{V}) \xrightarrow{i} F^{-1}(\mathcal{W}).$$

Then one can verify, using the above lemmas, that the new labeling gives a decomposition strategy for $\mathcal{X}$. □

We are now in the position to generalize Theorem 2.4.3 to finite decomposition complexity.

Theorem 2.8.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be families of metric spaces and let $F : \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map. If $\mathcal{Y}$ has finite decomposition complexity and for every bounded subspace $\mathcal{V}$ of $\mathcal{Y}$ the inverse image $F^{-1}(\mathcal{V})$ has finite decomposition complexity, then $\mathcal{X}$ has finite decomposition complexity.*

Proof. Consider a decomposition strategy for $\mathcal{Y}$ and, as in the previous proof, pull it back to obtain a labeled decomposition tree for $\mathcal{X}$. The difference with the previous situation is that the terminal vertices might now be labeled by families of the form $F^{-1}(\mathcal{V})$, which are not necessarily bounded families. However, each $F^{-1}(\mathcal{V})$ has finite decomposition complexity. Let $T_{F^{-1}(\mathcal{V})}$ denote the decomposition strategy for $F^{-1}(\mathcal{V})$. Attach $T_{F^{-1}(\mathcal{V})}$ to T at the vertex labeled by $F^{-1}(\mathcal{V})$. The resulting tree is a decomposition tree for $\mathcal{X}$. □

As before, the above theorem has several applications.

Corollary 2.8.6. *Let G be a countable discrete group acting by isometries on a metric space X with finite decomposition complexity. Suppose that there exists $x_0 \in X$ such that for every $R > 0$ the R-stabilizer of x_0 has finite decomposition complexity. Then G has finite decomposition complexity.*

Proof. We restrict to the orbit $O(x_0)$ of x_0. The map $G \to O(x_0)$ is surjective and equivariant, and the claim follows from Theorem 2.8.5. □

Corollary 2.8.7. *The class of groups with finite decomposition complexity is closed under extensions.*

Another consequence of Theorem 2.8.5 is the following generalization of Proposition 2.7.4.

Theorem 2.8.8. *Let X be a bounded geometry metric space with finite asymptotic dimension. Then X has finite decomposition complexity.*

The proof requires a theorem of Dranishnikov [70], see also [78], stating that a bounded geometry metric space embeds coarsely into a Cartesian product of finitely many trees. Since trees have asymptotic dimension 1, they also have finite decomposition complexity. Applying Theorem 2.8.5 we conclude that a product of finitely many trees also has finite decomposition complexity and, consequently, any space that coarsely embeds into such a product.

2.9 Groups with finite decomposition complexity

Decomposition complexity was modeled on the notion of asymptotic dimension, but it is much more general and flexible. In the previous sections we discussed examples of groups and metric spaces which do not have finite asymptotic dimension. These examples do however have finite decomposition complexity.

Proposition 2.9.1. *The group $\mathbb{Z}^{(\infty)}$, defined in Example 2.6.1, has finite decomposition complexity.*

Proof. We play the decomposition game on $\mathbb{Z}^{(\infty)} = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$. Let player B give a number R_0. Let $n \geq R_0$ and let $H = \mathbb{Z}^n$, generated by the coordinates indexed by $0, \dots, n-1$.

We then take the decomposition

$$\mathcal{X}_i = H_i,$$

where H_i are cosets of H. This gives an R_0-decomposition of $\mathbb{Z}^{(\infty)}$ over H_i. The claim follows from the fact that each H_i has finite asymptotic dimension. □

We now give an example of a finitely generated group with infinite asymptotic dimension but finite decomposition complexity. Recall that a Laurent polynomial p with integer coefficients satisfies

$$p(\pi) = \sum_{k=-l}^{l} n_k \pi^k,$$

where $n_k \in \mathbb{Z}$. Consider the group

$$G = \left\{ \begin{pmatrix} \pi^n & p(\pi) \\ 0 & \pi^{-n} \end{pmatrix} \mid n \in \mathbb{Z},\ p \text{ is a Laurent polynomial with coefficients in } \mathbb{Z} \right\}.$$

G is a subgroup of $\mathrm{SL}(2, \mathbb{R})$. The subgroup $H \subseteq G$, defined by

$$H = \left\{ h \in G \mid h = \begin{pmatrix} 1 & p(\pi) \\ 0 & 1 \end{pmatrix} \right\},$$

is isomorphic to the group $\mathbb{Z}^{(\infty)}$. We remark that $\mathbb{Z} \wr \mathbb{Z}$ is a subgroup of G, consisting of all matrices with the right upper entry to be of the form $p(\pi^2)$. Hence $\operatorname{asdim} G = \infty$.

Proposition 2.9.2. *The group G has finite decomposition complexity.*

Proof. The claim follows from the short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 0$$

and Theorem 2.8.5. □

A similar strategy shows that elementary amenable groups have finite decomposition complexity. We will discuss amenability in the next section, however the class of elementary amenable groups can be described without referring to amenability. Namely, this class is the smallest class containing finite and abelian groups and which is closed under taking quotients, subgroups, extensions and directed unions. The proof again uses Theorem 2.8.5. The Grigorchuk group is a certain amenable group which is not in the class of elementary amenable groups [102].

Open question 2.9.3. *Does Grigorchuk's group have finite decomposition complexity?*

The following result shows that the class of groups with finite decomposition complexity is very large.

Theorem 2.9.4. *Let $\mathbb{F}$ be a field and let G be a countable subgroup of* $\mathrm{GL}(n, \mathbb{F})$. *Then G has finite decomposition complexity.*

We refer to [117] for a proof.

Exercises

Exercise 2.1. Show, using only the definition, that asdim $\mathbb{Z}^2$ is *exactly* 2.

Exercise 2.2. Show that the Banach spaces $\ell_p(\mathbb{N})$ (or any Banach space which does not have a finite basis) have infinite asymptotic dimension.

Exercise 2.3. Prove the following characterization: a metric space X satisfies asdim $X \leq n$ if and only if for every $R > 0$ there exist families of sets $\mathcal{U}_0, \ldots, \mathcal{U}_n$ which together give a covering of X, all elements $\mathcal{U}_i$ are uniformly bounded in diameter and every $\mathcal{U}_i$ is R-disjoint, i.e., for any $U, V \in \mathcal{U}_i$ we have $d(U, V) \geq R$.

Exercise 2.4. State and prove a Lie group analogue of Theorem 2.4.3.

Exercise 2.5. Show that the wreath product of finitely generated groups is finitely generated.

Exercise 2.6. Generalize Proposition 2.6.3 to the following statement: *if G and H both contain an element of infinite order, then* asdim $G \wr H = \infty$.

Notes and remarks

Topological dimension theory was founded in the twenties of the XXth century and since then has become a fundamental tool in geometric topology. The first monograph on the subject was written by Hurewicz and Wallman [134]. A detailed contemporary account of topological dimension theory is Engelking's book [84].

Asymptotic dimension was introduced by Gromov [108]. Its first major applications were found soon after in [248], [249]. In particular, in [249] the Novikov Conjecture was proved for groups having finite asymptotic dimension. Since then the theory of asymptotic dimension and its variations has flourished and by now there is a wealth of results concerning asymptotic dimension. We have omitted some of them, such as e.g. union theorems or permanence properties of asymptotic dimension under free products. The reader is referred to [30] and the references therein for more on these topics. We also refer to [253] for an overview of applications of methods such as asymptotic dimension and finite decomposition complexity to rigidity properties of manifolds.

Theorem 2.3.3 in [210] is proved in greater generality, including certain hyperbolic metric spaces. Another way to generalize Proposition 2.3.1 is to consider CAT(0) cube complexes in which cells are Euclidean cubes glued together by isometric identifications along the edges, and which satisfy non-positive curvature conditions, see e.g. [37]. It is a result of Wright [244] that finite-dimensional CAT(0) cube complexes (i.e., the ones with an upper bound on the dimensions of cells) have finite asymptotic dimension.

There are various refinements of asymptotic dimension which involve prescribing the relation between the growth of R (in R-multiplicity) and the uniform bound on the diameters of the cover. These quantitative versions of dimension are usually only quasi-isometry invariants and not coarse invariants. The reader will find more in [30], [48].

The estimate on the dimension in the fibering theorem above is not very efficient. A more precise counterpart of the product upper bound was proved in [76]; see also [29]. The estimate of dimension there is much finer, in particular one obtains there the expected formula $\operatorname{asdim} X \times Y \leq \operatorname{asdim} X + \operatorname{asdim} Y$.

Other examples of groups with finite asymptotic dimension include certain relatively hyperbolic groups [190], Coxeter groups [76] and mapping class groups [33].

The wreath products which are our examples of groups with infinite asymptotic dimension are finitely generated but not finitely presented; that is, the group can never be written in terms of finitely many generators subject to finitely many relations. There are many examples of groups which are finitely presented and have infinite asymptotic dimension. An interesting example is Thompson's group F, which is finitely presented and contains $\mathbb{Z}^n$ as a subgroup for every $n \in \mathbb{N}$; see [51].

Chapter 3
Amenability

This chapter is devoted to introducing amenability. The notion of amenability was defined by J. von Neumann in [179] and is now fundamental in many areas of mathematics, since it allows us to average over infinite groups.

3.1 Følner conditions

The notion of amenability of groups was introduced by von Neumann, who identified lack of amenability as the reason behind the Banach–Tarski paradox. Amenability admits many characterizations, several of which will be very useful for us. The starting point will be the geometric definition due to Følner. It allows one to define amenability in terms of an *isoperimetric condition*. Isoperimetry is the comparison of the volume of a set with the area of its boundary. Given a finitely generated group G with a word length metric and a subset $A \subseteq G$, define the R-boundary of A to be the set

$$\partial_R A = \{g \in G \setminus A \mid d(g, A) \leq R\}.$$

The isoperimetric condition describes amenability in terms of existence of finite sets with small R-boundary.

Definition 3.1.1. A finitely generated group G is *amenable* if for every $R > 0$ and $\varepsilon > 0$ there exists a finite set $F \subseteq G$ such that

$$\frac{\#\partial_R F}{\#F} \leq \varepsilon.$$

We will refer to the sets satisfying the above condition as *Følner sets*. We remark that one obtains an equivalent condition by fixing $R \geq 1$, however, for technical reasons flexibility of R will become important.

Example 3.1.2. A finite group is amenable. Indeed, since G is finite, we can choose $F = G$. In this case the boundary is empty,

$$\frac{\#\partial F}{\#F} = 0,$$

and F is a Følner set for any $R > 0$ and $\varepsilon > 0$.

Example 3.1.3. The group $\mathbb{Z}$ is amenable. Take $\varepsilon = 1/n$ for some $n \in \mathbb{N}$, let $R > 0$ and consider $F = B(0, nR) \subseteq \mathbb{Z}$. Then $\partial_R F = \{-R(n+1), \dots, -Rn-1\} \cup \{Rn+1, \dots, R(n+1)\}$ and we have

$$\frac{\#\partial_R F}{\#F} = \frac{2R}{2Rn+1} \leq \frac{1}{n}.$$

The same principle as in Example 3.1.3 applies to finitely generated abelian groups: balls form a sequence of Følner sets.

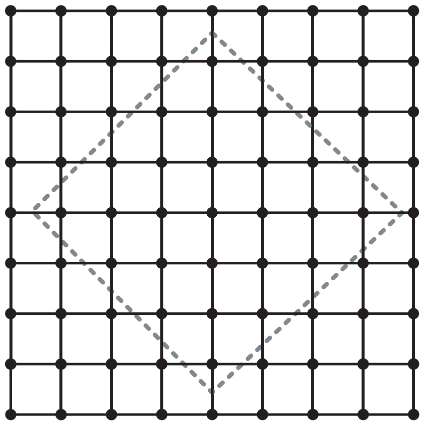

Figure 3.1. A ball of appropriately large radius is an example of a Følner set in $\mathbb{Z}^2$. In this case $\#F = 28$ and $\#\partial_1 F = 16$.

There are many examples of amenable groups. It is known that amenability is preserved under taking subgroups and quotients (see Proposition 3.1.4 below). It is also known that extensions of amenable groups by amenable groups, and directed unions of amenable groups are amenable. Thus any group constructed from finite and abelian groups by the above group constructions is again amenable. Such groups are often called *elementary amenable*, and it turns out that it is very difficult to construct amenable groups which are not elementary amenable. The question of whether such groups exist was asked by M. Day, and the first construction of an amenable but not elementary amenable group is due to R. Grigorchuk (see [102]).

The simplest example of a non-amenable group is a free group $\mathbb{F}_n$; we give a proof of its non-amenability in Example 3.3.3. Note that in terms of the isoperimetric condition, non-amenability of a group G means that cardinalities of boundaries of finite sets are comparable to the cardinalities of the set itself, i.e., there exists a $\delta > 0$ such that, for all finite sets $F \subseteq G$,

$$\frac{\#\partial_R F}{\#F} \geq \delta.$$

We now prove that amenability is inherited by subgroups. We prove this fact for finitely generated subgroups, although it holds in greater generality. One of the facts we need is that amenability is independent of the choice of the generating set, see Exercise 3.5.

Proposition 3.1.4. *Let G, H be finitely generated groups such that H is a subgroup of G. If G is amenable then H is amenable.*

Proof. Since amenability does not depend on the choice of the generating set, given a generating set Σ_H of H we can choose a generating set Σ_G of G that contains Σ_H. Let $R > 0$, $\varepsilon > 0$ and let F be the corresponding Følner set in G. Consider the set of left cosets $\{H_i\}_{i\in\mathbb{N}}$ of H in G and write

$$F_i = F \cap H_i$$

for every $i \in I$. In other words, F_i is the section of F along H_i. If we denote by ∂_R^G and ∂_R^H the R-boundary with respect to Σ_G and Σ_H, respectively, then for the set F we have

$$\#\partial_R^G F \geq \#\partial_R^H F = \sum_{i\in I} \#\partial_R^H F_i.$$

This yields

$$\varepsilon \geq \frac{\#\partial_R^G F}{\#F} \geq \sum_{i\in I} \frac{\#\partial_R^H F_i}{\#F} = \sum_{i\in I} \frac{\#F_i}{\#F} \cdot \frac{\#\partial_R^H F_i}{\#F_i}.$$

Observe that there must exist an index $i \in I$ such that

$$\frac{\#\partial_R^H F_i}{\#F_i} \leq \varepsilon.$$

Indeed, if we assume the contrary, then the equality $\sum \frac{\#F_i}{\#F} = 1$ contradicts the above inequality. Since all H_i are isometric, F_i is a Følner set in H for R and ε. □

The geometric nature of amenability has an important consequence, namely invariance under large scale equivalences.

Theorem 3.1.5. *Let G and H be quasi-isometric finitely generated groups. If H is amenable then G is amenable.*

Proof. Let $f : G \to H$ be a quasi-isometry with constants $L, C > 0$. Without loss of generality we may assume that f is onto, since otherwise we can find a finite group Γ and a map $f : G \times \Gamma \to H$ which is an onto quasi-isometry (possibly with different constants), as in Exercise 1.11. Then it is enough to show that $G \times \Gamma$ is amenable since G, as a subgroup, will also be amenable.

Fix $R > 0$ and $\varepsilon > 0$ and consider a Følner set $F_H \subseteq H$ such that

$$\frac{\#\partial_{LR+C} F_H}{\# F_H} \leq \varepsilon.$$

Define

$$F_G = f^{-1}(F_H) \subseteq G.$$

The set F_G is finite and we will show that F_G is a Følner set in G for ε and R. Observe that the R-boundary of F_G is mapped by f inside the $(LR+C)$-boundary of F_H. We have

$$\#\partial_R F_G \leq \#\partial_{LR+C} F_H.$$

Since $\# F_H \leq \# F_G$, we obtain the inequality

$$\frac{\#\partial_R F_G}{\# F_G} \leq \frac{\#\partial_{LR+C} F_H}{\# F_H} \leq \varepsilon,$$

as asserted. $\square$

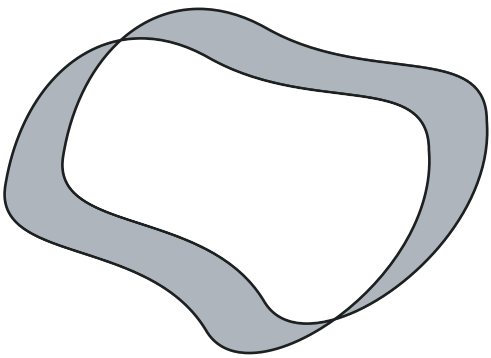

Figure 3.2. Følner condition as expressed in Proposition 3.1.6. The shaded area is the set $F \triangle \gamma F$.

We now give a different version of the isoperimetric condition. Let $F\gamma$ denote the set $\{g\gamma \mid g \in F\}$, the right translation of F by γ.

Proposition 3.1.6. *A finitely generated group is amenable if and only if for every* $R > 0$ *and* $\varepsilon > 0$ *there exists a finite set* $F \subseteq G$ *such that*

$$\frac{\#(F \triangle F\gamma)}{\# F} \leq \varepsilon$$

for every $\gamma \in G$ *with* $|\gamma| \leq R$.

Proof. Let us first prove a general estimate relating cardinalities of boundaries and sets of the form $F \triangle F\gamma$. Let $R > 0$, $\varepsilon > 0$ and $F \subseteq G$ be any finite set. Observe that

$$\bigcup_{|\gamma|\le R} F\gamma \setminus F = \partial_R F.$$

This gives

$$\#\partial_R F \le \sum_{|\gamma|\le R} \#(F\gamma \setminus F) \le \sum_{|\gamma|\le R} \#(F\gamma \triangle F). \tag{3.1}$$

For every $|\gamma| \le R$ we have

$$\#(F\gamma \triangle F) = \#(F\gamma \setminus F) + \#(F \setminus F\gamma) = \#(F \setminus F\gamma^{-1}) + \#(F \setminus F\gamma).$$

Since $|\gamma^{-1}| = |\gamma|$, we have

$$\begin{aligned}\sum_{|\gamma|\le R} \#(F\gamma \triangle F) &= 2\Big(\sum_{|\gamma|\le R} \#(F\gamma \setminus F)\Big)\\ &\le 2\#B(e,R) \max_{|\gamma|\le R} \#(F\gamma \setminus F)\\ &\le 2\#B(e,R)\#\partial_R F.\end{aligned}$$

The above, together with (3.1), gives the estimate

$$\frac{1}{2\#B(e,R)} \sum_{|\gamma|\le R} \#(F\gamma \triangle F) \le \#\partial_R F \le \sum_{|\gamma|\le R} \#(F\gamma \triangle F). \tag{3.2}$$

Let now $R > 0$ and $\varepsilon > 0$. If the group is amenable, then, by Definition 3.1.1, we have a finite set $F \subseteq G$ which satisfies

$$\frac{\#\partial_R F}{\#F} \le \frac{\varepsilon}{2\#B(e,R)}.$$

Then (3.2) implies that

$$\frac{\#(F\gamma \triangle F)}{\#F} \le 2\#B(e,R)\frac{\#\partial F}{\#F} \le \varepsilon.$$

The other direction follows similarly from (3.2). □

There is also a version of the above criterion for left translation.

Proposition 3.1.7. *A finitely generated group is amenable if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a finite set $F' \subseteq G$ such that*

$$\frac{\#(\gamma F' \triangle F')}{\#F'} \le \varepsilon,$$

for every $\gamma \in G$ with $|\gamma| \le R$.

Proof. Consider the inversion map[1] on G, $v\colon G \to G$ given by $v(g) = g^{-1}$. Let $\varepsilon > 0$, $R > 0$, and let $F \subseteq G$ be a set as in Proposition 3.1.6. Define $F' = v(F)$. Since v is a bijection and $\gamma F' = v(F\gamma^{-1})$, it is easy to check that F' satisfies the required condition. □

The geometric definition of amenability can be used effectively to show that certain groups are amenable.

Definition 3.1.8. Let G be a finitely generated group with a symmetric generating set Σ. The *volume function* of G associated to Σ, $\mathrm{Vol}_{G,\Sigma}\colon \mathbb{N} \to \mathbb{N}$, is defined by

$$\mathrm{Vol}_{G,\Sigma}(n) = \#B(e,n),$$

where G is equipped with the word length metric associated to Σ.

Note that the explicit values of Vol depend on the choice of the generating set, but the growth type (i.e., exponential, polynomial of a given degree, etc.) does not, and we will usually omit the reference to Σ. A finitely generated group has at most exponential volume growth; that is, for such a group there exists a constant $C > 0$ such that

$$\mathrm{Vol}_G(n) \le \exp(Cn)$$

for all $n \in \mathbb{N}$. We say that G has *subexponential volume growth* if

$$\lim_{n\to\infty} \frac{\ln \mathrm{Vol}_G(n)}{n} = 0.$$

If the above limit does not vanish, then G is said to be of *exponential growth*.

Example 3.1.9. Let $G = \mathbb{Z}^k$ with the standard generating set. Then G has polynomial growth for any $k \in \mathbb{N}$. More precisely,

$$\mathrm{Vol}_G(n) = \frac{1}{k!}(2n)^k + \frac{1}{(k-1)!}(2n)^{k-1} + \text{lower order terms}.$$

Example 3.1.10. Let $G = \mathbb{F}_k$ be the free group on $k \ge 2$ generators, with the standard generating set. Then G has exponential growth. More precisely,

$$\mathrm{Vol}_G(n) = \frac{k}{k-1}(2k-1)^n - \frac{1}{k-1}.$$

Growth of a group is an invariant of quasi-isometries (see Exercise 3.11) and is related to amenability.

Proposition 3.1.11. *Let G have subexponential volume growth. Then G is amenable.*

[1] We remark that the inversion map is not an isometry, it only preserves the distance to the origin.

Proof. Consider the ball $B(e,n)$ centered at the identity of radius n in G. One can decompose it into disjoint sets,

$$B(e,n) = B(e,n-1) \cup S(e,n) = B(e,n-1) \cup \partial_1 B(e,n),$$

where $S(e,n)$ denotes the sphere of radius n centered at the identity. Thus we can write

$$\begin{aligned}
\#B(e,n) &= \#B(e,n-1) + \#S(e,n) \\
&= \#B(e,n-1)\Big(1 + \frac{\#S(e,n)}{\#B(e,n-1)}\Big) \\
&= \#B(e,n-2)\Big(1 + \frac{\#S(e,n-1)}{\#B(e,n-2)}\Big)\Big(1 + \frac{\#S(e,n)}{\#B(e,n-1)}\Big) \\
&= \cdots \\
&= \prod_{k=1}^{n} \Big(1 + \frac{\#S(e,k)}{\#B(e,k-1)}\Big).
\end{aligned}$$

If G is non-amenable, then

$$\Big(1 + \frac{\#S(k)}{\#B(k-1)}\Big) \geq 1 + C > 0$$

for every $k \in \mathbb{N}$. By the above calculations this implies exponential growth. □

Examples of groups to which the above example applies are all finitely generated abelian groups, which have polynomial growth. Groups which have a nilpotent subgroup of finite index, i.e., virtually nilpotent groups, also have polynomial growth. Moreover, Gromov's famous Polynomial Growth Theorem states that the converse is true: *a group which has polynomial volume growth must be virtually nilpotent*. There are also groups whose volume growth function grows faster than any polynomial but slower than any exponential function. Such groups are said to have *intermediate volume growth*. Examples were constructed by Grigorchuk [101], as an answer to a question of Milnor. We refer to the survey article [104] and to [65] for an excellent introduction to Grigorchuk's constructions and a thorough treatment of growth of groups, as well as to [103] for a detailed historical background.

3.2 The Hulanicki–Reiter condition

Recall from Example 1.1.7 that given a discrete set X we write

$$\ell_1(X) = \Big\{ f : X \to \mathbb{R} \mid \sum_{x \in X} |f(x)| \leq \infty \Big\},$$

with the corresponding metric defined in Example 1.1.7. We also write

$$\ell_1(X)_{1,+} = \{f \in \ell_1(X) \mid \|f\|_1 = 1 \text{ and } f \geq 0\}.$$

In other words, $\ell_1(X)_{1,+}$ is the set of non-negative probability measures on X. In the case when $X = G$ is a group, the natural action of G on itself induces an action on the set of functions on G, in particular on $\ell_1(G)$, by the formula

$$\gamma \cdot f(g) = f(g\gamma)$$

for all $f \in \ell_1(G)$ and $\gamma \in G$. It is straightforward to verify that this is indeed an action of G. The next lemma will be necessary for both amenability and later for property A since it will allow us to pass between the geometric and functional descriptions.

Lemma 3.2.1. *Let $f \in \ell_1(X)_{1,+}$ be a finitely supported function and let $\varepsilon > 0$. There exists a number $N > 0$ and a function $\tilde{f} \in \ell_1(X)_{1,+}$ such that $\tilde{f} : X \to \{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1\}$ and $\|f - \tilde{f}\|_1 \leq \varepsilon$.*

The proof is left as an easy exercise. The next statement gives a useful analytic description of amenability.

Theorem 3.2.2 (Hulanicki–Reiter condition). *A finitely generated group G is amenable if and only if for every $\varepsilon > 0$ and $R > 0$ there exists a function $f \in \ell_1(G)_{1,+}$ such that*

(1) *$\|f - \gamma \cdot f\|_1 \leq \varepsilon$ for all $\gamma \in G$ satisfying t $|\gamma| \leq R$, and*

(2) *$\operatorname{supp} f$ is finite.*

Proof. Assume that the group is amenable. By Proposition 3.1.6, for any $\varepsilon > 0$ and $R > 0$ there exists a Følner set $F \subseteq G$ satisfying

$$\frac{\#(F \bigtriangleup F\gamma)}{\#F} \leq \varepsilon$$

whenever $|\gamma| \leq R$. Define

$$f = \frac{1_F}{\#F}.$$

It is clear that $\|f\|_1 = 1$ and that condition (2) holds. To verify condition (1) we observe that

$$\begin{aligned}\|f-\gamma\cdot f\|_1 &= \sum_{g\in G}|f(g)-f(g\gamma)| \\ &= \sum_{g\in G}\frac{|1_F(g)-1_F(g\gamma)|}{\#F} \\ &= \frac{1}{\#F}\sum_{g\in G}|1_F(g)-1_{F\gamma^{-1}}(g)| \\ &= \frac{\#(F\triangle F\gamma^{-1})}{\#F} \;\leq\; \varepsilon.\end{aligned}$$

To prove the other direction let $\varepsilon > 0$, $R > 0$, and let $f \in \ell_1(G)_{1,+}$ be a finitely supported function satisfying $\|f-\gamma\cdot f\|_1 \leq \delta$, where the dependence of δ on ε will be chosen at the end of the proof. As in Lemma 3.2.1 we approximate f by $\tilde{f}$ for some number N so that

$$\|\tilde{f}-f\|_1 \leq \delta.$$

Then

$$\|\tilde{f}-\gamma\cdot\tilde{f}\|_1 \leq \|\tilde{f}-f\|_1 + \|f-\gamma\cdot f\|_1 + \|\gamma\cdot f-\gamma\cdot\tilde{f}\|_1 \leq 3\delta$$

whenever $|\gamma| \leq R$. We now define a finite set $A \subseteq G\times\mathbb{N}$ by the relation

$$(g,i)\in A \quad \text{if and only if} \quad \frac{i}{N} \leq \tilde{f}(g).$$

Note that $\#A = N$ and we estimate

$$\sum_{|\gamma|\leq R}\frac{\#(A\gamma\triangle A)}{\#A} = \sum_{|\gamma|\leq R}\|\tilde{f}-\gamma\cdot\tilde{f}\|_1 \leq \#B(e,R)3\delta, \tag{3.3}$$

where $A\gamma = \{(g\gamma,i)\in G\times\mathbb{N} \mid (g,i)\in A\}$. Set $A^i = A\cap(G\times\{i\})$ so that A^i is the horizontal section at the level i of the set A. Then we have

$$\sum_{|\gamma|\leq R}\frac{\#(A\gamma\triangle A)}{\#A} = \sum_{i\in\mathbb{N}}\sum_{|\gamma|\leq R}\frac{\#(A^i\gamma\triangle A^i)}{\#A} = \sum_{i\in\mathbb{N}}\frac{\#A^i}{\#A}\Big(\sum_{|\gamma|\leq R}\frac{\#(A^i\gamma\triangle A^i)}{\#A^i}\Big). \tag{3.4}$$

We claim that there exists an index $k\in\mathbb{N}$ such that

$$\sum_{|\gamma|\leq R}\frac{\#(A^k\gamma\triangle A^k)}{\#A^k} \leq 3\delta\#B(e,R).$$

Indeed, assume the contrary, that is, for all $i\in\mathbb{N}$ we have

$$\sum_{|\gamma|\leq R}\frac{\#(A^i\gamma\triangle A^i)}{\#A^i} > 3\delta\#B(e,R).$$

Then by (3.4) we have

$$\sum_{|\gamma|\leq R} \frac{\#(A\gamma \triangle A)}{\#A} > \sum_{i\in\mathbb{N}} \frac{\#A^i}{\#A}(3\delta\#B(e,R)) = 3\delta\#B(e,R)$$

since $\sum_i \#A^i = \#A$. This however contradicts (3.3). To finish the proof, define $F = \{g \mid (g,k) \in A^k\}$, note the inequality

$$\frac{\#(Fh \triangle F)}{\#F} \leq \sum_{|\gamma|\leq R} \frac{\#(F\gamma \triangle F)}{\#F}$$

whenever $|h| \leq R$, and choose $\delta = \varepsilon/3\#B(e,R)$. Then F satisfies the conditions of Proposition 3.1.6 for the given ε and R. □

The function f, as in the above theorem, will be called a Hulanicki–Reiter function for R and ε.

Remark 3.2.3. The second condition in the above theorem is not necessary in the formulation since finitely supported functions are dense in $\ell_1(G)$.

3.3 Invariant means

There is one more characterization of amenability which will be crucial from our point of view. It involves the notion of an invariant mean.

The normalized Haar measure on a compact group G is an invariant probability measure on G and averaging using this measure is a useful tool in different settings. However, on an infinite discrete group the Haar measure is just the counting measure and it is never finite. To obtain an invariant probability measure on an infinite discrete group we have to give up countable additivity. This leads to the notion of an invariant mean.

Definition 3.3.1. Let X be a set. A *mean* on $\ell_\infty(X)$ is a continuous linear functional $M : \ell_\infty(X) \to \mathbb{R}$ which satisfies

(1) $M(f) \geq 0$ whenever $f \geq 0$,

(2) $M(1_X) = 1$,

for every $f \in \ell_\infty(X)$.

If $X = G$ is a group, then we say that the mean is *invariant* if additionally

(3) $M(\gamma \cdot f) = M(f)$

holds for every $f \in \ell_\infty(G)$ and $\gamma \in G$.

An invariant mean can be thought of as integration against a *finitely additive* probability measure which is invariant under the translation action of G. In fact, the formula $\mu(A) = M(1_A)$ for a set $A \subseteq G$, defines such a measure. On an infinite group, the measure μ cannot be described explicitly. For example, note that the measure of any point, and consequently of every finite set, is zero. However, μ can be described as a certain limit in the space of all means on $\ell_\infty(G)$. We use this idea to prove the following important statement.

Theorem 3.3.2. *A finitely generated group G is amenable if and only if there exists an invariant mean on $\ell_\infty(G)$.*

Proof. Let $f_n \in \ell_1(X)_{1,+}$ be a Hulanicki–Reiter function for $\varepsilon = 1/n$ and $R = 1$. In particular,

$$\lim_{n\to\infty} \|f_n - \gamma \cdot f_n\|_1 = 0$$

for every $\gamma \in G$ with $|\gamma| \le 1$. Consider the functional $M_n \colon \ell_\infty(G) \to \mathbb{R}$, induced by f_n in the following way:

$$M_n(\varphi) = \sum_{g\in G} \varphi(g) f_n(g)$$

for any $\varphi \in \ell_\infty(G)$. In other words, M_n is the discrete integration against the probability measure f_n. It follows from the definition that M_n is a mean on $\ell_\infty(G)$, in particular an element of $\ell_\infty(G)^*$, the dual space of $\ell_\infty(G)$. As elements of $\ell_\infty(G)^*$, all M_n are of norm 1 and thus are all contained in the closed unit ball in $\ell_\infty(G)^*$. We now apply the Banach–Alaoglu theorem, which states that the unit ball in a dual space is compact when equipped with the weak*-topology (see [201], [214], [245]). This means that there is a subnet $\{M_\beta\}$ which converges in the weak*-topology. Define

$$M = w^*\text{-}\lim_\beta M_\beta,$$

where $w^*\text{-}\lim$ denotes the limit in the weak* topology. The limit is a mean (this is easy to check) and we need to prove invariance under the group action. It is enough to prove invariance under the generators of the group. Let $\sigma \in G$ be a generator and observe that

$$\begin{aligned}
|M(\varphi) - M(\sigma\cdot\varphi)| &= |\lim_\beta M_\beta(\varphi) - \lim_\beta M_\beta(\sigma\cdot\varphi)| \\
&\le \lim_\beta |M_\beta(\varphi) - M_\beta(\sigma\cdot\varphi)| \\
&\le \lim_\beta \|\varphi\|_\infty \, \|f_\beta - \sigma^{-1}\cdot f_\beta\|_1 \\
&= 0.
\end{aligned}$$

We only sketch the converse direction. Recall that $\ell_1(G)$ is dense in its double dual, $\ell_\infty(G)^*$, equipped with the weak-* topology. Thus we can approximate a mean by a net of finitely supported means f_β in $\ell_1(G)$ which are asymptotically invariant in the weak topology on $\ell_1(G)$. To guarantee asymptotic invariance in norm, we use the Mazur lemma, which implies that there exists a sequence of finite linear combinations of ν_β that is asymptotically invariant in norm. $\square$

The existence of an invariant mean on an amenable group allows one to give a proof of the fact that free groups $\mathbb{F}_n$, $n \geq 2$, are not amenable.

Example 3.3.3. The free group $\mathbb{F}_2$ is not amenable. We will show that there is no invariant mean on $\mathbb{F}_2$. Assume the contrary and denote by a and b the generators of $\mathbb{F}_2$. Let A the set of all words in $\mathbb{F}_2$ which begin with a non-zero power of a. Observe that

$$A \cup aA = \mathbb{F}_2.$$

This implies that $M(1_A) \geq \frac{1}{2}$. However the sets $A, bA, b^2A, \ldots$ are all disjoint and

$$M(1_A) = M(1_{b^k A})$$

for every $k \in \mathbb{N}$. Thus

$$1 = M(1_{\mathbb{F}_2}) \geq M(1_A) + M(1_{bA}) + M(1_{b^2A}) \geq \frac{3}{2},$$

and the claim is proved.

By Proposition 3.1.4 we obtain

Corollary 3.3.4. *Every finitely generated group which has a free subgroup is not amenable.*

The characterization of amenability in terms of invariant means has another important consequence.

Corollary 3.3.5. *Let G be amenable and let $N \subseteq G$ be a normal subgroup of G. Then the quotient group G/N is amenable.*

Proof. It is enough to prove that there is an invariant mean on G/N. Observe that the space $\ell_\infty(G/N)$ is isometrically isomorphic to the space of those functions in $\ell_\infty(G)$ which are constant on each coset of N. Thus an invariant mean on G restricts to an invariant mean on G/N. $\square$

Exercises

Exercise 3.1. Compute $\frac{\#\partial_R F}{\#F}$ for $F = B(e,n) \subseteq \mathbb{Z}^2$, depending on n and R.

Exercise 3.2. Show that we obtain an equivalent definition of amenability if we consider $\partial_R F = \{x \in F \mid d(x, G \setminus F) \leq R\}$.

Exercise 3.3. Show that we obtain an equivalent definition of amenability if we consider $\partial_R F = \{x \in G \mid d(x, F) \leq R$ and $d(x, G \setminus F) \leq R\}$.

Exercise 3.4. Prove that we get an equivalent of amenability if we replace "for every $R > 1$" by "for some $R > 1$" in Definition 3.1.1.

Exercise 3.5. Show that amenability of a finitely generated group does not depend on the choice of the generating set.

Exercise 3.6. Prove that for the free group $\mathbb{F}_2$ with its standard generating set the inequality $\frac{\#\partial_1 F}{\#F} \geq 1$ hold for every finite set $F \subseteq \mathbb{F}_2$.
(Hint: First prove that $\frac{\#\partial_1 F}{\#F} \geq 2$ for any finite subtree F. Then use this fact to treat the general case.)

Exercise 3.7. Use the Hulanicki–Reiter condition to show that $\mathbb{F}_2$ is not amenable.

Exercise 3.8. Show that amenability is preserved under taking direct products; that is, if G and H are both amenable then $G \times H$ is also amenable.

Exercise 3.9. Use the Hulanicki–Reiter condition to show that if G is amenable and $H \leq G$ is a subgroup of G, then H is amenable.

Exercise 3.10. Show that amenability is a quasi-isometry invariant, using the Hulanicki–Reiter property.

Exercise 3.11. Given two functions $f, f' \colon [0,\infty) \to [0,\infty)$ we say that f' grows faster than f, written $f \preceq f'$, if there exist constants $C, D > 0$ such that $f(t) \leq Cf'(Dt)$ for all sufficiently large t. We say that f and f' have the same growth, written $f \simeq f'$, if $f \preceq f'$ and $f' \preceq f$. Show that if G and H are finitely generated groups, then their volume functions have the same growth.

Notes and remarks

As mentioned already in the introduction, amenability was introduced by von Neumann in 1929 in the context of the Banach–Tarski paradox. The reader interested in this aspect of amenability is referred to [216], [233]. Other important references on amenability are [100], [200], [203]. Also [26], Appendix G, covers the introductory material. In particular the above references describe various applications of amenability.

It was a long-standing problem, known as the *von Neumann conjecture*, whether the existence of a free subgroup characterizes amenability. The answer is negative. Examples of non-amenable groups without free subgroups were constructed by Olshanskii [189]. A weaker, geometric version of the von Neumann conjecture was proved by Whyte [237], who showed that a group is not amenable if and only if it can be partitioned into copies of a tree. If G has a free subgroup, then the cosets of the subgroup give such a partition.

Amenability of a group is equivalent to vanishing of bounded cohomology groups of the group. This fact was first proved by B. E. Johnson in terms of vanishing of the continuous Hochschild cohomology of the algebra $\ell_1(G)$, with coefficients in any dual $\ell_1(G)$-bimodule [137]. This allowed the definition of amenability of a Banach algebra via vanishing of its first cohomology group. Amenability of Banach algebras is an active area of research, see [216], [200].

A homological characterization of amenability in terms of vanishing of a certain homology group is due to Block and Weinberger [34], [35]. This result will be discussed in Chapter 7.

Chapter 4
Property A

Property A is a metric property and can be viewed as a weak version of amenability. It was introduced in [250] and is a sufficient condition for coarse embeddability into a Hilbert space, discussed in Chapter 5.

4.1 Definition and basic properties

We begin with the definition.

Definition 4.1.1. Let X be a uniformly discrete metric space. We say that X has *property A* if for every $\varepsilon > 0$ and $R > 0$ there exists a collection of finite subsets $\{A_x\}_{x\in X}$, $A_x \subseteq X \times \mathbb{N}$ for every $x \in X$, and a constant $S > 0$ such that

(1) $\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} \leq \varepsilon$ when $d(x,y) \leq R$, and
(2) $A_x \subseteq B(x,S) \times \mathbb{N}$.

The above conditions mean that the sets A_x and A_y are almost equal when $d(x,y) \leq R$ and are disjoint if $d(x,y) \geq 2S$. Note also that condition (1) above resembles the condition characterizing amenability in Proposition 3.1.6. The copy of $\mathbb{N}$ appearing in the definition is necessary for technical reasons: it allows us to count points of X with multiplicities, which is needed to prove invariance under coarse equivalences.

Example 4.1.2. Let G be a finite group. Define $A_x = G \times \{1\}$. Then it is easy to check that

$$\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} = 0$$

and clearly both conditions are satisfied. Thus finite groups have property A.

Example 4.1.3. The group $\mathbb{Z}$ has property A. To see this, take $\varepsilon = 1/n$, $R > 0$ and define $A_x = B(x, R(n+1)) \times \{1\}$. Then if $d(x,y) \leq R$ we have $\#(A_x \triangle A_y) \leq 2R$ while $\#(A_x \cap A_y) \geq 2Rn$, and the claim follows.

In the above examples the construction of the sets A_x is similar to the construction of Følner sets for $\mathbb{Z}$. This is not a coincidence.

Proposition 4.1.4. *Let G be a finitely generated group. If G is amenable then G has property A.*

Proof. Let $\varepsilon > 0$, $R > 0$ and let $\delta = \frac{\varepsilon}{1+\varepsilon}$. Let F be a Følner set for the left translation action, corresponding to δ and R, as in Proposition 3.1.7. Define

$$A_g = gF \times \{1\}.$$

Let $d(g,h) \leq R$. We have

$$\begin{aligned}\frac{\#(A_g \triangle A_h)}{\#(A_g \cap A_h)} &= \frac{\#(gF \triangle hF)}{\#(gF \cap hF)} \\ &= \frac{\#(F \triangle g^{-1}hF)}{\#(F \cap g^{-1}hF)} \\ &= \frac{\#(F \triangle g^{-1}hF)}{\#F} \cdot \frac{\#F}{\#(F \cap g^{-1}hF)}.\end{aligned} \tag{4.1}$$

By the properties of F, the first factor in the above product is bounded above by δ, and we will show that the second factor is close to 1. For any sets A, B, the sets $A \cap B$ and $A \triangle B$ are disjoint, and $A \cup B = (A \triangle B) \cup (A \cap B)$. We obtain

$$\frac{\#(F \cap g^{-1}hF)}{\#F} = \frac{\#(F \cup g^{-1}hF)}{\#F} - \frac{\#(F \triangle g^{-1}hF)}{\#F} \geq 1 - \delta. \tag{4.2}$$

Since $|g^{-1}h| \leq R$, it follows from (4.1) and (4.2) that

$$\frac{\#(A_g \triangle A_h)}{\#(A_g \cap A_h)} \leq \frac{\delta}{1-\delta} = \varepsilon.$$

□

Note that a counterpart of condition (2) of Definition 4.1.1 was implicit in the definition of amenability since we only had to deal with a single finite set and its translates, which were automatically disjoint if translated far enough.

We have seen that the free group $\mathbb{F}_n$ is not amenable, nor is any subgroup which contains a free subgroup (by Exercise 3.9). However, it turns out that free groups do have property A.

Example 4.1.5. We will show that an infinite tree T has property A. Choose a root $x_0 \in T$ and fix a geodesic ray ω starting at x_0 (existence of such a ray is guaranteed by König's lemma). For any other point $x \in T$ there exists a unique geodesic ray ω_x such that $\omega \cap \omega_x$ is infinite. Given $\varepsilon = 2/n$ and $R > 0$ we take the set $A_x \subseteq T \times \{1\}$ to be the vertices on the geodesic segment of length $(n+1)R$ on the ray ω_x, starting at x. Then $A_x \subseteq B(x, R(n+1)) \times \{1\}$, and if $d(x,y) \leq R$ then

$$\#(A_x \triangle A_y) \leq 2R$$

while

$$\#(A_x \cap A_y) \geq nR.$$

The quotient is bounded above by ε and it follows that a tree has property A. Since the Cayley graph of any free group is a tree, free groups have property A.

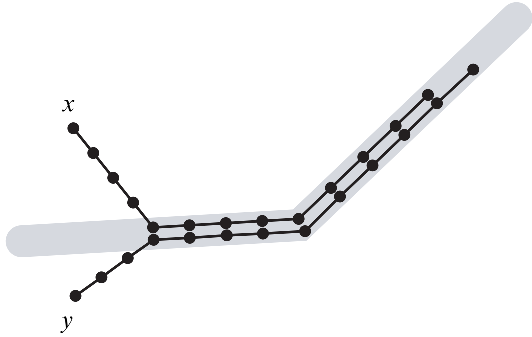

Figure 4.1. Property A for a tree. The thick gray line is the geodesic ray ω. In this case $\#(A_x \bigtriangleup A_y) = 8$ and $\#(A_x \cap A_y) = 9$. As ω_x and ω_y go further along ω we obtain arbitrarily small ratios since $\#(A_x \bigtriangleup A_y) = 8$ while $\#(A_x \cap A_y) \to \infty$.

We will show later in this chapter that hyperbolic groups also have property A. Since most hyperbolic groups are not amenable (they have free subgroups), we can see that property A is a much broader notion than amenability. The question how much broader is not easy to answer. Although not all metric spaces and groups have property A, examples are hard to find. The last section in this chapter is devoted to constructing examples of metric spaces which do not have property A. For groups the question is much more difficult. At present there is one construction, due to Gromov [109], [110], of groups which for certain geometric reasons cannot have property A. We refer to [7] for details. It is an open problem how to construct more explicit examples.

Just as with amenability, property A is a large scale invariant.

Theorem 4.1.6 ([250]). *Let X and Y be coarsely equivalent, uniformly discrete metric spaces with bounded geometry. If Y has property A then X has property A as well.*

Proof. We will prove the result in the case when X and Y are quasi-isometric and leave the general case of coarse equivalence as an exercise. Let $R > 0$, $\varepsilon > 0$ and let $f : X \to Y$ be an (L, C)-quasi-isometry. Let B_y be a family of sets giving property A for Y, for ε/N and $LR + C$, where $N = \sup_{w \in Y} \# f^{-1}(w)$. For every $x \in X$ define a set $A_x \subseteq X \times \mathbb{N}$,

$$A_x = \{(z, n) \in X \times \mathbb{N} \mid (f(z), n) \in B_{f(x)}\}.$$

In other words, we pull-back the vertical sections of $B_{f(x)}$ to vertical sections of A_x. It is easy to check that the sets A_x are finite and that there exists a number $S' > 0$ such that $A_x \subseteq B(x, S') \times \mathbb{N}$. Note also that

$$\#(A_x \bigtriangleup A_y) \leq N \cdot \#(B_{f(x)} \bigtriangleup B_{f(y)}),$$

where $x, y \in X$ such that $d_X(x, y) \leq R$. We have

$$d_Y(f(x), f(y)) \leq LR + C,$$

and it follows that

$$\frac{\#(A_x \triangle A_y)}{\#(B_{f(x)} \cap B_{f(y)})} \leq N(\varepsilon/N) = \varepsilon.$$

On the other hand,

$$\#(A_x \cap A_y) \geq \#(B_{f(x)} \cap B_{f(y)}),$$

and we obtain the inequality

$$\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} \leq \frac{\#(A_x \triangle A_y)}{\#(B_{f(x)} \cap B_{f(y)})}\varepsilon. \qquad \square$$

4.2 The Higson–Roe condition

We now prove a characterization parallel to the Hulanicki–Reiter condition in Theorem 3.2.2. The idea of the proof is similar as in the case of amenable groups, but we need some auxiliary facts.

Theorem 4.2.1 (Higson–Roe condition, [130]). *Let X be a uniformly discrete metric space with bounded geometry. X has property A if and only if for every $\varepsilon > 0$ and every $R > 0$ there exists a map $\xi \colon X \to \ell_1(X)_{1,+}$, $x \mapsto \xi_x$, and a number $S > 0$ such that*

(1) $\|\xi_x - \xi_y\| \leq \varepsilon$ *if* $d(x, y) \leq R$ *and*
(2) $\operatorname{supp} \xi_x \subseteq B(x, S)$.

The first lemma is a simple exercise in counting set elements.

Lemma 4.2.2. *For any finite sets A and B we have*

$$|\#A - \#B| \leq \#(A \triangle B).$$

The second lemma states that as ε becomes smaller, the sets A_x and A_y for nearby points $x, y \in X$ have almost equal cardinalities.

Lemma 4.2.3. *Let X be a metric space with property A and let $\{A_x\}$ and $S > 0$ satisfy the conditions of Definition 4.1.1 for a given $\varepsilon > 0$ and $R > 0$. If $d(x, y) \leq R$, then*

$$1 \leq \frac{\#A_x}{\#(A_x \cap A_y)} \leq 1 + \varepsilon \quad \textit{and} \quad \frac{1}{1+\varepsilon} \leq \frac{\#A_x}{\#A_y} \leq 1 + \varepsilon.$$

Proof. Let $d(x, y) \leq R$. Since for any sets A, B we have $A = (A \setminus B) \cup (A \cap B)$, the sets $A \setminus B$ and $A \cap B$ are disjoint and $A \setminus B \subseteq A \vartriangle B$, it follows that

$$\#(A_x \setminus A_y) \leq \varepsilon \#(A_x \cap A_y)$$

and

$$\#(A_x \cap A_y) \leq \#A_x = \#(A_x \setminus A_y) + \#(A_x \cap A_y) \leq (1 + \varepsilon)\#(A_x \cap A_y).$$

The first inequality in the statement of the lemma follows by dividing both sides by $\#(A_x \cap A_y)$. Exactly the same inequality holds for $\#A_y$ and by combining the two we obtain the second inequality. □

Lemma 4.2.3 gives a useful reformulation of property A.

Lemma 4.2.4. *A uniformly discrete metric space X has property A if and only if for every $R > 0$ and $\varepsilon > 0$ there exists a family $\{A_x\}_{x \in X}$ of finite subsets $A_x \subseteq X \times \mathbb{N}$, $x \in X$, and a number $S > 0$ such that $A_x \subseteq B(x, S) \times \mathbb{N}$ and*

$$\frac{\#(A_x \vartriangle A_y)}{\#A_x} \leq \varepsilon$$

for every $x, y \in X$ satisfying $d(x, y) \leq R$.

Proof. The "if" direction follows from the previous lemma and the equality

$$\frac{\#(A_x \vartriangle A_y)}{\#(A_x \cap A_y)} = \frac{\#(A_x \vartriangle A_y)}{\#A_x} \cdot \frac{\#A_x}{\#(A_x \cap A_y)}.$$

To prove the "only if" direction we proceed similarly:

$$\frac{\#(A_x \vartriangle A_y)}{\#A_x} = \frac{\#(A_x \vartriangle A_y)}{\#(A_x \cap A_y)} \cdot \frac{\#(A_x \cap A_y)}{\#A_x}$$

and since $(A_x \cap A_y) \subseteq A_x$, we obtain

$$\frac{\#(A_x \cap A_y)}{\#A_x} \leq 1.$$

This proves the assertion. □

Proof of Theorem 4.2.1. First we prove the "only if" direction. Let $\varepsilon > 0$ and $R > 0$. Given a corresponding family of sets $\{A_x\}$ and a number $S > 0$, guaranteed by Definition 4.1.1, we write $A_x^z = A_x \cap (\{z\} \times \mathbb{N})$ so that A_x^z is the vertical section of A_x at z. Define for every $x \in X$ a function $\xi_x \colon X \to \mathbb{R}$ by setting

$$\xi_x(z) = \frac{\#A_x^z}{\#A_x}.$$

First note that $\xi_x(z) \geq 0$ for every $x, z \in X$ and that

$$\sum_{z \in X} \xi_x(z) = \sum_{z \in X} \frac{\#A_x^z}{\#A_x} = 1$$

for every $x \in X$. Thus $\xi_x \in \ell_1(X)_{1,+}$ for every $x \in X$. It is also easy to see that $\operatorname{supp} \xi_x \subseteq B(x, S)$. To prove the last condition we observe that

$$\|\#A_x \xi_x - \#A_y \xi_y\|_1 = \sum_{z \in X} |\#A_x^z - \#A_y^z| \leq \sum_{z \in X} \#(A_x^z \bigtriangleup A_y^z) = \#(A_x \bigtriangleup A_y).$$

(Note that we used Lemma 4.2.2 in the first inequality.) The above calculation implies that

$$\left\| \xi_x - \frac{\#A_y}{\#A_x} \xi_y \right\|_1 \leq \frac{\#\left(A_x \bigtriangleup A_y\right)}{\#A_x} \leq \frac{\#\left(A_x \bigtriangleup A_y\right)}{\#(A_x \cap A_y)}$$

since $\#(A_x \cap A_y) \leq \#A_x$. Now, by Lemma 4.2.3, we have

$$\begin{aligned}
\|\xi_x - \xi_y\|_1 &\leq \left\| \xi_x - \frac{\#A_x}{\#A_y} \xi_y \right\|_1 + \left\| \frac{\#A_x}{\#A_y} \xi_y - \xi_y \right\|_1 \\
&\leq \frac{\#(A_x \bigtriangleup A_y)}{\#(A_x \cap A_y)} + \left| \frac{\#A_x}{\#A_y} - 1 \right| \cdot \|\xi_y\|_1 \\
&\leq 2\varepsilon.
\end{aligned}$$

Conversely, let $\varepsilon > 0$ and $R > 0$ and assume that functions ξ_x as above exist. By Lemma 3.2.1 we can replace each ξ_x by a function $\tilde{\xi}_x \in \ell_1(X)_{1,+}$ such that

$$\|\xi_x - \tilde{\xi}_x\|_1 \leq \varepsilon,$$

and each $\tilde{\xi}_x$ takes values in $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$. A priori N depends on x and can be different for every ξ_x. However, since X has bounded geometry and $\operatorname{supp} f \subseteq B(x, S)$ for a fixed S, then also $\# \operatorname{supp} f \leq \#B(x, S) \leq M$ for some $M > 0$. This implies that N can be chosen uniform for all ξ_x so that it depends only on ε. Now define the sets A_x by the formula

$$A_x = \left\{(y, i) \in X \times \mathbb{N} \mid \xi_x(y) \geq \tfrac{i}{N}\right\}.$$

For every $x \in X$ the set A_x is finite and $A_x \subseteq B(x, S) \times \mathbb{N}$. By construction, $\#A_x = N$ for every $x \in X$ and we have

$$\begin{aligned}
\frac{\#(A_x \bigtriangleup A_y)}{\#A_x} &= \frac{\#(A_x \bigtriangleup A_y)}{N} \\
&= \|\tilde{\xi}_x - \tilde{\xi}_y\|_1 \\
&\leq \|\tilde{\xi}_x - \xi_x\|_1 + \|\xi_x - \xi_y\|_1 + \|\xi_y - \tilde{\xi}_y\|_1 \\
&\leq 3\varepsilon.
\end{aligned}$$

Combining this with Lemma 4.2.4 we obtain the claim. □

We want to point out that the "only if" direction does not require the bounded geometry assumption.

It follows easily from the above theorem and Theorem 3.2.2 (the Hulanicki–Reiter condition) that amenability implies property A. This relation also illustrates the slogan that property A is a "non-equivariant" amenability. Indeed, if in the above theorem we would require the $\xi\colon G \to \ell_1(G)$ to be equivariant, then we would get *exactly* the conditions of Theorem 3.2.2.

Amenability is inherited by subgroups. Property A behaves similarly. The following fact was proved independently by Dranishnikov, and Januszkiewicz [76] and Tu [231]

Proposition 4.2.5. *Let X be a uniformly discrete metric space and let $Y \subseteq X$ be a subspace of X. If X has property A, then Y has property A as well.*

Proof. For every point $x \in X \setminus Y$ choose a point $p(x) \in Y$ satisfying

$$d(x, p(x)) \leq 2d(x, Y).$$

If we additionally define $p(y) = y$ for all $y \in Y$, then we obtain a function $p\colon X \to Y$, $x \mapsto p(x)$. Given the Higson–Roe function $\xi\colon X \to \ell_1(X)_{1,+}$ for $R > 0$ and $\varepsilon > 0$, as in Theorem 4.2.1, we define

$$\eta_y(z) = \sum_{x \in p^{-1}(z)} \xi_y(x)$$

for $y, z \in Y$. If $y, w \in Y$ satisfy $d(y, w) \leq R$, then

$$\begin{aligned}
\|\eta_y - \eta_w\|_1 &= \sum_{z\in Y} |\eta_y(z) - \eta_w(z)| \\
&= \sum_{z\in Y} \Big| \sum_{x\in p^{-1}(z)} \xi_y(x) - \xi_w(x) \Big| \\
&\leq \sum_{z\in Y} \sum_{x\in p^{-1}(z)} |\xi_y(x) - \xi_w(x)| \\
&\leq \sum_{x\in X} |\xi_y(x) - \xi_w(x)| \\
&\leq \|\xi_y - \xi_w\|_1 \\
&\leq \varepsilon.
\end{aligned}$$

To finish the proof note that

$$d(p(x), z) \leq d(p(x), x) + d(x, z) \leq 2d(x, Y) + d(x, z) \leq 3d(x, z)$$

for every $z \in Y$. It follows that $\operatorname{supp} \eta_y \subseteq B(y, 3S)$, where $S > 0$ is such that $\operatorname{supp} \xi_y \subseteq B(y, S)$. □

4.3 Finite asymptotic dimension implies property A

We have seen in the previous section that amenability implies property A. In this section we present a theorem due to Higson and Roe [130], relating property A and finite asymptotic dimension. We first recall the notion of partition of unity.

Definition 4.3.1. Let X be a topological space. A *partition of unity* is a family of continuous functions $\mathcal{P} = \{\varphi_i : X \to [0,1]\}$ such that for every $x \in X$ there is a neighborhood V of x for which $\varphi_i|_V \neq 0$ only for finitely many $i \in I$ and

$$\sum_{i \in I} \varphi_i(x) = 1$$

for every $x \in X$.

Let $\mathcal{U}$ be a cover of X. A partition of unity $\{\varphi_U\}_{U \in \mathcal{U}}$ is said to be *subordinate* to this cover if $\operatorname{supp} \varphi_U \subseteq U$ for every $U \in \mathcal{U}$.

Example 4.3.2. Consider an interval $[a,b] \subseteq \mathbb{R}$. The functions $\varphi_1, \varphi_2 : [a,b] \to [0,1]$, $\varphi_1(x) = \sin^2 x$, $\varphi_2(x) = \cos^2 x$ form a partition of unity on $[a,b]$.

Example 4.3.3. Let X be a metric space and let $\mathcal{U}$ be a locally finite cover of X. Define $\varphi_U : U \to [0,1]$ by the formula

$$\varphi_U(x) = \frac{d(x, X \setminus U)}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)}.$$

It is straightforward to verify that $\mathcal{P} = \{\varphi_U\}_{U \in \mathcal{U}}$ is a partition of unity subordinate to the cover $\mathcal{U}$. It has the additional property that $\operatorname{supp} \varphi_U = U$.

Lemma 4.3.4. *Let X be a metric space and $A \subseteq X$ any subset. Then*

$$|d(x,A) - d(y,A)| \leq d(x,y).$$

The proof relies on the triangle inequality and is left as an exercise. The following lemma describes a useful property of partitions of unity of the above form.

Lemma 4.3.5. *Let $\mathcal{V}$ be a uniformly bounded cover of X of multiplicity κ and Lebesgue number $2r$, where $r \geq 1$. Let $\{\varphi_V\}_{V \in \mathcal{V}}$ be the partition of unity subordinate to $\mathcal{V}$, described in Example 4.3.3. Then, for each $V \in \mathcal{V}$, the function φ_V is $(2\kappa + 1)r^{-1}$-Lipschitz.*

Proof. Since the Lebesgue number of $\mathcal{V}$ is at least $2r$, we have

$$\sum_{V \in \mathcal{V}} d(x, X \setminus V) \geq r$$

for every $x \in X$. Indeed, for every x the ball $B(x, R)$ has to be contained in some $V \in \mathcal{V}$, which implies that $d(x, X \setminus V) \geq r$.

We have

$$\begin{aligned}
&|\varphi_V(x) - \varphi_V(y)| \\
&= \left| \frac{d(x, X \setminus V)}{\sum_{V \in \mathcal{V}} d(x, X \setminus V)} - \frac{d(y, X \setminus V)}{\sum_{V \in \mathcal{V}} d(y, X \setminus V)} \right| \\
&\leq \frac{|d(x, X \setminus V) - d(y, X \setminus V)|}{\sum_{V \in \mathcal{V}} d(x, X \setminus V)} + \left| \frac{d(y, X \setminus V)}{\sum_{V \in \mathcal{V}} d(x, X \setminus V)} - \frac{d(x, X \setminus V)}{\sum_{V \in \mathcal{V}} d(y, X \setminus V)} \right| \\
&\leq \frac{d(x, y)}{\sum_{V \in \mathcal{V}} d(y, X \setminus V)} + \frac{d(y, X \setminus V)\big(\sum_{V \in \mathcal{V}} |d(y, X \setminus V) - d(x, X \setminus V)|\big)}{\big(\sum_{V \in \mathcal{V}} d(y, X \setminus V)\big)\big(\sum_{V \in \mathcal{V}} d(x, X \setminus V)\big)} \\
&\leq \frac{d(x, y)}{r} + \frac{1}{r^2}\Big(\sum_{V \in \mathcal{V}} |d(y, X \setminus V) - d(x, X \setminus V)|\Big).
\end{aligned}$$

Since $|d(y, X \setminus V) - d(x, X \setminus V)| \neq 0$ only if $x \in V$ or $y \in V$, using Lemma 4.3.4 we obtain

$$\sum_{V \in \mathcal{V}} |d(y, X \setminus V) - d(x, X \setminus V)| \leq \sum_{V \in \mathcal{V}(x,y)} d(x, y),$$

where $\mathcal{V}(x, y) = \{V \in \mathcal{V} \mid x \in V \text{ or } y \in V\}$. Clearly, $\#\mathcal{V}(x, y) \leq 2\kappa$ since the multiplicity of the cover is κ. This yields

$$|\varphi_V(x) - \varphi_V(x)| \leq \left(\frac{1}{r} + \frac{2\kappa}{r}\right) d(x, y) \leq \frac{2\kappa + 1}{r} d(x, y). \qquad \square$$

Note that we used the assumption $r \geq 1$ in the form of the inequality $r^{-2} \leq r^{-1}$.

Theorem 4.3.6. *Let X be a uniformly discrete metric space such that* $\operatorname{asdim} X < \infty$*. Then X has property A.*

Proof. The idea of the proof is to modify slightly the covers provided by the definition of asymptotic dimension, take an associated partition of unity and construct the functions ξ_x, using this partition.

Let $R > 0$ and $\mathcal{U}$ be a uniformly bounded cover of X with $5R$-multiplicity $\kappa = \operatorname{asdim} X + 1$. Take a new cover,

$$\mathcal{V} = \{\mathrm{nbhd}_{2R}(U) \mid U \in \mathcal{U}\},$$

where $\mathrm{nbhd}_{2R}(U) = \{x \in X \mid d(U, x) \leq 2R\}$ is the $2R$-neighborhood of the set U. Note that the multiplicity of the cover $\mathcal{V}$ is again bounded by $\operatorname{asdim} X + 1$, and, additionally, the Lebesgue number of $\mathcal{V}$ is at least $2R$. Let now $\mathcal{P} = \{\varphi_V\}_{V \in \mathcal{V}}$ be a partition of unity subordinate to $\mathcal{V}$, described in Example 4.3.3. By the previous lemma, for every $V \in \mathcal{V}$ the function φ_V is $(2\kappa + 1)R^{-1}$-Lipschitz. For every $V \in \mathcal{V}$ choose a point $x_V \in V$ and define $\xi_x : X \to [0, 1]$ by the formula

$$\xi_x(y) = \begin{cases} \varphi_V(x) & \text{if } y = x_V, \\ 0 & \text{otherwise.} \end{cases}$$

We first need to check that these are in fact elements of $\ell_1(X)_{1,+}$. We have

$$\sum_{y \in X} \xi_x(y) = \sum_{V \in \mathcal{V}} \varphi_V(x) = 1$$

for every $x \in X$.

As for the support condition, since the multiplicity of $\mathcal{V}$ equals $\kappa = \operatorname{asdim} X + 1$, only κ of φ_V are not zero at any $x \in X$, and $\operatorname{supp} \xi_x$ is contained in the union of those V's which contain x. Thus $\operatorname{supp} \xi_x \subseteq B(x, D)$, where $D = \sup_{V \in \mathcal{V}} \operatorname{diam} V$ is finite by construction and the conditions on $\mathcal{U}$.

Finally, we estimate

$$\begin{aligned} \|\xi_x - \xi_y\|_1 &= \sum_{z \in X} |\xi_x(z) - \xi_y(z)| \\ &= \sum_{V \in \mathcal{V}} |\varphi_V(x) - \varphi_V(y)| \\ &\leq \sum_{V \in \mathcal{V}(x,y)} d(x, y) \frac{2\kappa + 1}{R} \\ &\leq d(x, y) \frac{(2\kappa + 1)(2\kappa)}{R}. \end{aligned}$$

(The notation $\mathcal{V}(x, y)$ here is carried over from the previous proof.) It remains to choose R sufficiently large to obtain the appropriate ε for ξ. The existence of such R for a fixed κ is guaranteed by finiteness of the asymptotic dimension. □

We have proved in Chapter 2 that hyperbolic groups have finite asymptotic dimension. This fact, together with the above theorem, leads to

Corollary 4.3.7. *Let G be a hyperbolic group. Then G has property A.*

We remark that there is no relation between finite asymptotic dimension and amenability. For instance the wreath products constructed in Section 2.6 are amenable and have infinite asymptotic dimension, while free groups have asymptotic dimension 1 but, as we have seen, are not amenable.

Finite decomposition complexity implies asymptotic property C, which in turn implies property A [68] (the proof of the latter implication is essentially the same as the argument in the proof of Theorem 4.3.6).

Theorem 4.3.8 ([118]). *Let X be a uniformly discrete metric space with finite decomposition complexity. Then X has property A.*

The same argument gives a similar statement for weak finite decomposition complexity.

Remark 4.3.9. A generalization of Theorem 4.3.6 to certain groups with infinite asymptotic dimension is possible. In [72] Dranishnikov studied groups which have infinite asymptotic dimension, but the growth rate of multiplicities of certain uniformly bounded covers of the group can be controlled by a polynomial function. It was shown that such groups have property A. This argument was later extended by Ozawa [198], showing that subexponential control is sufficient to conclude property A. In general, such dimension growth functions are at most exponential for any finitely generated group. In [77] Dranishnikov and Sapir showed that the Thompson group has dimension growth of order at least $e^{\sqrt{n}}$.

4.4 Property A and residually finite groups

We have seen several examples of spaces which satisfy property A: amenable groups, trees, free and hyperbolic groups, spaces with finite asymptotic dimension. A natural question arises: what are examples of spaces that do not have property A? In this section we construct two such examples. The first construction is based on properties of residually finite groups and has bounded geometry.

Definition 4.4.1. A finitely generated group G is called *residually finite* if there exists a sequence of normal subgroups $N_i \leq G$ of finite index such that $\bigcap_{i\in\mathbb{N}} N_i = \{e\}$.

Example 4.4.2. The group $\mathbb{Z}^n$ is residually finite for any $n \in \mathbb{N}$ since we can choose $N_i = i\,\mathbb{Z}^n$.

The metric spaces we will be interested in are coarse disjoint unions of finite quotient groups G/N_i. Each such quotient is a finite metric space, with the metric induced by the images of generators of G under the quotient map. For instance, for the group $\mathbb{Z}$ and a sequence of subgroups $i\,\mathbb{Z}$ the finite quotients are the finite cyclic groups $\mathbb{Z}_i$. These spaces also have bounded geometry, this is left as one of the exercises. Our first observation is that the geometry of the finite quotients approximates the geometry of the group G.

Proposition 4.4.3. *Let G be a residually finite group and let $N_i \subseteq G$ be a sequence of finite index subgroups with trivial intersection such that $N_i \subseteq N_j$ whenever $j \leq i$. Then for every $R > 0$ there exists $i_0 \in \mathbb{N}$ such that $B_G(e, R)$ is isometric to $B_{G/N_i}(e, R)$ for all $i \geq i_0$.*

The proof is left to the reader. We will need the following quantitative relation between amenability and property A.

Theorem 4.4.4. *Let G be an amenable group, let $\varepsilon > 0$ and $R > 0$. Then for any $S > 0$ the following conditions are equivalent:*

(1) *there exists a map $\xi\colon G \to \ell_1(G)_{1,+}$ such that $\|\xi_x - \xi_y\|_1 \le \varepsilon$ when $d(x, y) \le R$ and $\operatorname{supp} \xi_x \subseteq B(x, S)$;*
(2) *there exists a function $f \in \ell_1(G)_{1,+}$ such that $\|f - \gamma \cdot f\|_1 \le \varepsilon$ when $|\gamma| \le R$ and $\operatorname{supp} f \subseteq B(e, S)$.*

The emphasis in this theorem is on the constant S. More precisely, if the Higson–Roe condition is satisfied with $\varepsilon > 0$ and $R > 0$, with the diameter of the support of ξ at most $2S$, then the Hulanicki–Reiter function for the same ε and R can be chosen to have the diameter of the support at most $2S$.

Proof. We already know how to show that (2) implies (1): given the Hulanicki–Reiter function $f \in \ell_1(G)_{1,+}$ for $\varepsilon > 0$ and $R > 0$, define $\xi_g = g \cdot f$ for all $g \in G$. It is straightforward to check that ε, R and S are preserved.

To prove the other direction take a Higson–Roe function $\xi\colon G \to \ell_1(G)_{1,+}$ for $\varepsilon > 0$ and $R > 0$. Since G is amenable, there exists a mean M on $\ell_\infty(G)$ which is invariant in the sense that

$$M(\gamma \cdot f) = M(f)$$

for $f \in \ell_\infty(G)$, $\gamma \in G$, where $\gamma \cdot f(h) = f(h\gamma)$.

For every $h \in G$ define

$$f(h) = M(\xi_g(gh)),$$

where the averaging is with respect to g. The function $f : G \to \mathbb{R}$ is well defined since $\xi_g(h) \le 1$ for all $g, h \in G$ and $\xi_g(gh)$, as a function of g, belongs to $\ell_\infty(G)$.

First note that if $d(g, gh) = |h| > S$, then $\xi_g(gh) = 0$ for all $g \in G$. Thus $f(h) = 0$ whenever $|h| > S$. Consequently,

$$\begin{aligned}\|f\|_1 &= \sum_{h \in B(e,S)} f(h) = \sum_{h \in B(e,S)} M(\xi_g(gh)) \\ &= M\Big(\sum_{h \in B(e,S)} \xi_g(gh)\Big) = M(1_G) = 1.\end{aligned}$$

In other words, $f \in \ell_1(G)_{1,+}$.

For the purposes of the next computation denote the left translation action by $(\gamma \bullet f)(h) = f(\gamma^{-1}h)$ for $f : G \to \mathbb{R}$ and $\gamma, h \in G$. If $\gamma \in G$ is such that $|\gamma| \le R$, then

$$\begin{aligned}
\|f - \gamma \bullet f\|_{\ell_1(G)} &= \sum_{h \in G} |f(h) - f(\gamma^{-1}h)| \\
&= \sum_{h \in B(e,S) \cup \gamma B(e,S)} |M(\xi_g(gh)) - M(\xi_g(g\gamma^{-1}h))| \\
&= \sum_{h \in B(e,S) \cup \gamma B(e,S)} |M(\xi_g(gh)) - M(\xi_{g\gamma}(gh))| \\
&= \sum_{h \in B(e,S) \cup \gamma B(e,S)} |M(\xi_g(gh) - \xi_{g\gamma}(gh)))| \\
&\leq M\Big(\sum_{h \in B(e,S) \cup B(\gamma,S)} |\xi_g(gh) - \xi_{g\gamma}(gh)|\Big) \\
&\leq M(\varepsilon 1_G) = \varepsilon.
\end{aligned}$$

In the third equality in the sequence above we have used the invariance of the mean as follows:

$$M(\xi_g(g\gamma^{-1}h)) = M(\xi_{(g\gamma)}((g\gamma)\gamma^{-1}h)) = M(\xi_{g\gamma}(gh)).$$

Thus for the previously chosen R and ε we have constructed a function $f \in \ell_1(G)_{1,+}$ satisfying $\|f - \gamma \bullet f\|_{\ell_1(G)} \leq \varepsilon$ whenever $|\gamma| \leq R$ and $\operatorname{supp} f \subseteq B(e,S)$ for the same S as for ξ. Applying the inversion map $g \mapsto g^{-1}$ to f we obtain a Hulanicki–Reiter function with the same parameters, as required in the theorem. $\square$

The following lemma will allow us to express property A for a disjoint union in terms of a local condition.

Lemma 4.4.5. *Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of finite metric spaces. The coarse disjoint union of X_i has property A if and only if for every $\varepsilon > 0$, $R > 0$ and for all but finitely many $i \in \mathbb{N}$, the space X_i has a Higson–Roe function ξ^i for ε and R satisfying $\operatorname{supp} \xi^i_g \subseteq B(x,S)$, where $S > 0$ independent of i.*

The proof is left as Exercise 4.2. We are now ready to give the first construction of metric spaces which do not have property A. They were constructed in [209] and are known as *box spaces*.

Theorem 4.4.6 ([209]). *Let G be a residually finite group and let $N_i \subseteq G$, $i \in \mathbb{N}$, be normal, finite index subgroups of G satisfying $N_i \subseteq N_j$ if $i \geq j$ and $\bigcap_{i \in \mathbb{N}} N_i = \{e\}$. The coarse disjoint union of the finite metric spaces $\{G/N_i\}_{i \in \mathbb{N}}$ has property A if and only if G is amenable.*

Proof. Assume first that G is amenable and let $\varepsilon > 0$, $R > 0$. There exists a Hulanicki–Reiter function $f \in \ell_1(G)_{1,+}$ such that

$$\|f - \gamma \cdot f\|_1 \leq \varepsilon$$

for $|\gamma| \leq R$, and there is an $S > 0$ such that $\operatorname{supp} f \subseteq B(e, S)$. By Proposition 4.4.3 there is an index i_0 such that $B_G(e, 2(S+R))$ is isometric to $B_{G/N_i}(e, 2(S+R))$ for all $i \geq i_0$. Let $\varphi_i : B_{G/N_i}(e, 2(S+R)) \to B_G(e, 2(S+R))$ be this isometry. For all these i we define $f_i \in \ell_1(G/N_i)_{1,+}$ to be

$$f_i(g) = \begin{cases} f(\varphi_i(g)) & \text{if } g \in B_{G/N_i}(e, 2(S+R)) \text{ and } i \geq i_0, \\ 0 & \text{if } g \notin B_{G/N_i}(e, 2(S+R)) \text{ and } i \geq i_0, \\ \frac{1}{\#G/N_i} & \text{if } i < i_0. \end{cases}$$

It is easy to check that f_i is a Hulanicki–Reiter function on G/N_i for ε and R. Also,

$$\sup_{i \in \mathbb{N}} \operatorname{diam}(\operatorname{supp} f_i) < \infty,$$

and the fact that the coarse disjoint union has property A follows from Theorem 4.4.4.

We prove the converse by reversing the above argument. Assume that the coarse disjoint union has property A, that is, for a given $\varepsilon > 0$, $R > 0$ it satisfies the Higson–Roe condition. By Lemma 4.4.5, this is equivalent to the Higson–Roe condition for G/N_i for every $i \in \mathbb{N}$, with the same ε, R and $S > 0$ which is independent of i. Since every G/N_i is finite, it is also amenable and, by the averaging theorem for property A (Theorem 4.4.4), we deduce the existence of functions $f_i \in \ell_1(G/N_i)_{1,+}$ satisfying

$$\|f - \gamma \cdot f\|_1 \leq \varepsilon$$

whenever $|\gamma| \leq R$ and $\operatorname{supp} f_i \subseteq B(e, S)$ for every $i \in \mathbb{N}$. There exists an index $i \in \mathbb{N}$ for which the ball $B_{G/N_i}(e, 2(S+R))$ is isometric to $B_G(e, 2(S+R))$, via an isometry φ_i as above. Define

$$f(g) = \begin{cases} f_i(\varphi_i^{-1}(g)) & \text{if } g \in B_G(e, 2(S+R)), \\ 0 & \text{if } g \notin B_G(e, 2(S+R)). \end{cases}$$

Then f is a Hulanicki–Reiter function for ε and R. □

For instance, for the group $\mathbb{Z}$ and a sequence of its subgroups $2^n\mathbb{Z} \leq \mathbb{Z}$, the coarse disjoint union has property A since $\mathbb{Z}$ is amenable. However, we are mainly interested in the opposite situation.

Example 4.4.7. The free group $\mathbb{F}_2$ is residually finite (see [154]) and non-amenable. Choose any sequence of normal, finite index subgroup $N_i \subseteq \mathbb{F}_2$ such that $N_{i+1} \subseteq N_i$ and $\bigcap_{i=1}^{\infty} N_i = \{e\}$. Then the coarse disjoint union $\coprod \mathbb{F}_2/N_i$ does not have property A.

Let us discuss another important example. Consider the group $\mathrm{SL}(2,\mathbb{Z})$, the group of (2×2)-matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a,b,c,d \in \mathbb{Z}$ and $ad-bc=1$. This group is not amenable since the subgroup $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ generated by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is free. This can be proved using the so-called *ping-pong lemma*, see e.g. [65].

The group $\mathrm{SL}(2,\mathbb{Z})$ is residually finite. The congruence subgroups N_m of $\mathrm{SL}(2,\mathbb{Z})$ are defined as the kernel of the map $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})$. They have finite index in $\mathrm{SL}(2,\mathbb{Z})$ and also satisfy $\bigcap N_i = \{e\}$. By Theorem 4.4.6 we have the following

Corollary 4.4.8. *Let $p \geq 2$ be any prime number. The coarse disjoint union of* $\mathrm{SL}(2,\mathbb{Z}_{p^n})$*, $n = 1, 2, \ldots$, does not have property A.*

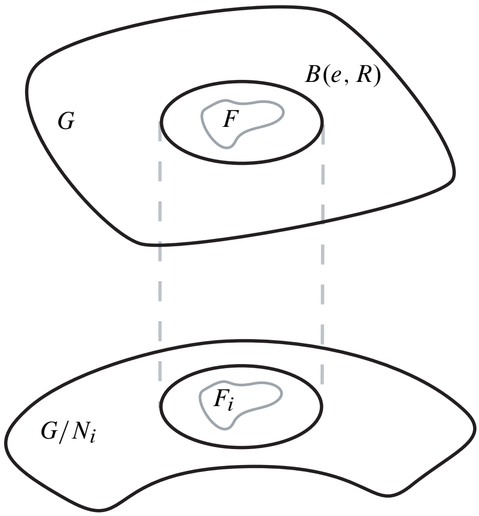

Figure 4.2. Mapping the Hulanicki–Reiter functions between G and its quotient G/N_i. Since the balls of radius R are isometric, we can transfer the Hulanicki–Reiter functions within them.

A beautiful generalization of Theorem 4.4.6 for the coarse disjoint unions of graphs which are not necessarily Cayley graphs was proved by Willett. Recall that the girth of a graph X, denoted by $g(X)$, is the length of the shortest cycle in X.

Theorem 4.4.9 ([239]). *Let $\{X_i\}_{i=1}^{\infty}$ be a collection of graphs such that every vertex x of X_i satisfies $3 \leq \deg(x) \leq k$ for any $i \in \mathbb{N}$. If $\lim_{i\to\infty} g(X_i) = \infty$, then the coarse disjoint union of X_i does not have property A.*

4.5 Locally finite examples

We now present the second construction of spaces without property A. In this construction we lose one feature that the previous examples enjoyed, namely bounded geometry. The spaces we construct now are locally finite but have an additional property which will be discussed in the next chapter: they can be naturally embedded coarsely into a Hilbert space. We need to examine the behavior of amenability and property A under direct products. For this purpose we introduce a quantitative measure of property A.

Definition 4.5.1. For an amenable group G let

$$S(G;\varepsilon) = \inf\{S \in \mathbb{Z} \mid \operatorname{supp} f \subseteq B_G(e,S)\},$$

where f runs through all Hulanicki–Reiter functions for $R = 1$ and $\varepsilon > 0$ in G.

In other words, $S(G;\varepsilon)$ measures the smallest possible diameter of supports of the Hulanicki–Reiter function for the given $\varepsilon > 0$ and $R = 1$.

Given a collection of groups G_i, $i = 1,\dots,n$, generated by finite sets Σ_i respectively, the product $G = \prod G_i$ is generated by $\Sigma = \bigcup_i \{e\} \times \cdots \times \{e\} \times \Sigma_i \times \{e\} \times \cdots \times \{e\}$. The corresponding metric on G can be described as

$$d_G(g,h) = \sum_{i=1}^{n} d_{G_i}(g_i,h_i),$$

where $g = (g_1,\dots,g_n)$ and $h = (h_1,\dots h_n)$ are elements of G.

Theorem 4.5.2. *Let G be a non-trivial, finitely generated amenable group and $0 < \varepsilon < 2$. Then*

$$\lim_{n\to\infty} S(G^n;\varepsilon) = \infty.$$

Proof. Assume the contrary. Then there exists $S \in \mathbb{N}$ and an infinite set $I \subseteq \mathbb{N}$ such that for every $i \in I$ there is a function $f_i \in \ell_1(G^i)_{1,+}$ satisfying

$$\|f_i - s \cdot f_i\|_1 \leq \varepsilon$$

and $\operatorname{supp} f_i \subseteq B_{G^i}(e,S)$ for all generators $s \in G$. The idea of the proof is to show that we can perform certain operations on the functions f_i in such a way that the functions obtained afterwards will have supports of smaller diameter while we still control the ε.

Let $\delta > 0$. We will determine the dependence of δ on ε at the end of the proof. Fix $m \in \mathbb{N}$ and for $i \in I$ consider the decomposition of the cartesian product

$$G^i = \underbrace{G^m \times \cdots \times G^m}_{k \text{ times}} \times G^r,$$

where r is the "remainder" satisfying $r < k$ and $mk + r = i$. We index the k consecutive factors G^m above by numbers $1, 2, \ldots, k$ and consider the set

$$\partial_S^j G^i = \{g \in G^i \mid d(g,e) = S \text{ and } g = (e, \ldots, e, \underbrace{\alpha}_{j\text{-th coordinate}}, e, \ldots, e)\}.$$

Let $\partial_S^j f_i \colon G^i \to \mathbb{R}$ be the restriction of f_i to the set $\partial_S^j G^i$.

We clearly have

$$\partial_S^j G^i \cap \partial_S^l G^i = \emptyset$$

whenever $j \neq l$ and

$$\sum_{j=1}^{k} \|\partial_S^j f_i\|_1 \le \|f_i\|_1 = 1.$$

It follows that for any prescribed $\hat{\varepsilon} > 0$ we can choose i sufficiently large (while keeping m fixed) so that there exists an index j for which

$$\|\partial_S^j f_i\| \le \hat{\varepsilon}.$$

We choose $\hat{\varepsilon} > 0$ so that the inequality

$$\frac{\varepsilon + 2\hat{\varepsilon}}{1 - \hat{\varepsilon}} \le \varepsilon + \delta$$

is satisfied and write

$$h = \frac{f_i - \partial_S^j f_i}{\|f_i - \partial_S^j f_i\|_1}.$$

Then $h \in \ell_1(G^i)_{1,+}$ and $\|f_i - \partial_S^j f_i\|_1 \ge 1 - \hat{\varepsilon}$. If s is a generator of G^i, we have the following estimate:

$$\begin{aligned}
\|h - s \cdot h\|_1 &= \frac{\|(f_i - s \cdot f_i) - (\partial_S^j f_i - s \cdot \partial_S^j f_i)\|_1}{\|f_i - \partial_S^j f_i\|_1} \\
&\le \frac{\|f_i - s \cdot f_i\|_1 + 2\|\partial_S^j f_i\|_1}{\|f_i - \partial_S^j f_i\|_1} \\
&\le \frac{\varepsilon + 2\hat{\varepsilon}}{1 - \hat{\varepsilon}} \le \varepsilon + \delta
\end{aligned}$$

by our previous choice of $\hat{\varepsilon}$. In other words, since $\partial_S^n f_i$ is very small, the ε-invariance under translation by generators also does not change much if we remove $\partial_S^j f_i$ from f_i. We now view

$$G^i = G^m \times G^{i-m},$$

where the first factor is the same factor in which we have performed the above "cropping" procedure. We define a function $f_m' \colon G^m \to \mathbb{R}$ by the formula

$$f_m'(\gamma) = \sum_{g \in G^{i-n}} h(\gamma, g).$$

In other words, f_m' is the projection of h onto the factor G^m. Note that $f_m' \in \ell_1(G^m)_{1,+}$ and that if s is a generator of G^m, then

$$\begin{aligned}
\|f_m' - s \cdot f_m'\|_1 &= \sum_{\gamma \in G^m} \Big| \sum_{g \in G^{i-m}} h(\gamma, g) - h(\gamma s, g) \Big| \\
&\le \sum_{\gamma \in G^m} \sum_{g \in G^{i-m}} |h(\gamma, g) - h(\gamma s, g)| \\
&= \sum_{\alpha \in G^i} |h(\alpha) - h(\alpha s)| \\
&= \|h - s \cdot h\|_1 \le \varepsilon + \delta.
\end{aligned}$$

However, note that $\operatorname{supp} f_m' \subseteq B_{G^m}(e, S-1)$ and recall that m is arbitrary. Thus we can repeat the procedure described above for every $m \in \mathbb{N}$ to obtain a function $\operatorname{supp} f_m' \subseteq B_{G^m}(e, S-1)$ satisfying

$$\|f_m' - s \cdot f_m'\|_1 \le \varepsilon + \delta.$$

We then apply the argument to these new functions and obtain a sequence of functions $f_m'' \in \ell_1(G^m)_{1,+}$, for which $\operatorname{supp} f_m'' \subseteq B_{G^m}(e, S-2)$ and

$$\|f_m'' - s \cdot f_m''\|_1 \le \varepsilon + 2\delta$$

for every generator $s \in G^m$. Then, after S repetitions, we obtain a sequence of functions $f_m^S \in \ell_1(G)_{1,+}$, for which $\operatorname{supp} f_m^{(S)} = \{e\}$ for every $m \in \mathbb{N}$. This means that

$$f_m^{(S)}(\gamma) = \begin{cases} 1 & \text{if } \gamma = e, \\ 0 & \text{otherwise,} \end{cases}$$

and we see that $\|f_m^{(S)} - s \cdot f_m^{(S)}\|_1 = 2$ for all generators $s \in G^m$. On the other hand,

$$\|f_m^S - s \cdot f_m^S\|_1 \le \varepsilon + S\delta.$$

Since δ was arbitrary, we can now choose it so that $\varepsilon + S\delta < 2$ to reach a contradiction. $\square$

The above estimate allows us to prove the main result of this section.

Theorem 4.5.3 ([183]). *Let G be a finite group of order at least 2. The coarse disjoint union of $\{G^n\}_{n\in\mathbb{N}}$ does not have property A.*

Proof. Assume the contrary. Then, by Lemma 4.4.5, property A implies the existence of Higson–Roe functions $\xi_n\colon G^n \to \ell_1(G^n)_{1,+}$, for each $n \in \mathbb{N}$, satisfying the conditions of Proposition 4.2.1 for $\varepsilon > 0$ and $R = 1$, with $S > 0$, independent of $n \in \mathbb{N}$. By Theorem 4.4.4, this is equivalent to existence of Hulanicki–Reiter functions $f_n \in \ell_1(G^n)_{1,+}$ for the same $\varepsilon > 0$ and $R = 1$, so that supp $f_n \subseteq B_{G_n}(e, S)$. However, by Theorem 4.5.2, this yields a contradiction. $\square$

The simplest case is when $G = \mathbb{Z}_2$. In this case G, as a metric space, can be simply viewed as two vertices $\{0, 1\}$ with distance 1 between them. The space G^n for each $n \in \mathbb{N}$ is then a cube with the so-called Hamming distance. By Theorem 4.5.3, the coarse disjoint union of such cubes does not have property A.

Open question 4.5.4. *Construct new examples of groups which do not satisfy property A.*

Exercises

Exercise 4.1. Let X and Y be uniformly discrete metric spaces with property A. Show that the Cartesian product $X \times Y$, with the metric $d((x, y), (x', y')) = (d(x, x')^p + d(y, y')^p)^{1/p}$, has property A for any $1 \le p < \infty$.

Exercise 4.2. Prove Lemma 4.4.5.

Exercise 4.3. Let G be a residually finite group and let $N_i \subseteq G$ be a sequence of finite index normal subgroups such that $N_i \le N_j$ whenever $i \le j$. Show that the coarse disjoint union of G/N_i has bounded geometry.

Exercise 4.4. Let X be a bounded geometry space of subexponential volume growth. Show that X has property A.

Notes and remarks

Permanence properties of groups with property A are studied by several authors. In particular, in [56] it is proved that property A is preserved by free products; in [63] it is proved that property A is preserved by direct limits and extensions; in [44], [185] it is proved that property A is preserved under taking quotients by amenable subgroups. Different characterizations and permanence properties are also proved in [231]. We also recommend [187] and the extensive survey [238].

For groups, property A turned out to be equivalent to various generalized forms of amenability. The Higson–Roe condition, discussed in this chapter, can be interpreted as amenability of the action of a group G on its Stone–Čech compactification. This is the same as saying that the groupoid of this action is amenable. See [6], [130] for details on amenable actions and groupoids.

Guentner and Kaminker [114], and Ozawa [195] characterized property A via a property of the reduced C*-algebra of a group. Namely, they showed that exactness of the reduced group C*-algebra of G is equivalent to property A for G. It also follows that property A is equivalent to nuclearity of the uniform Roe algebra (see [209]). See also [5] for a discussion of the above properties of C*-algebras.

Other applications of the averaging theorem for property A are in [182], [184], [185]. These include connections to isoperimetric invariants of finitely generated groups and to finer invariants related to asymptotic dimension.

Quantitative versions of property A were studied in [95], [182], [229]. They are also related to quantitative versions of asymptotic dimension, compression of coarse embeddings and other invariants of groups.

The fact that the finite quotients of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, discussed earlier do not have property A has a much deeper reason, namely that the Cayley graphs of these quotients form a sequence of expanding graphs. We will discuss expanders in the next section, see also [132], [152], [220] for an excellent introduction.

A different class of spaces without property A is the class of *warped cones*, constructed by Roe [211]. These spaces share some properties with the coarse disjoint union of quotients of residually finite groups constructed above.

Property A was proved for many interesting classes of groups. In particular, for relatively hyperbolic groups [197], mapping class groups [122], [147], linear groups [113], groups acting on CAT(0)-cube complexes [39], [50], groups acting on buildings [49].

Property A for groups was characterized in terms of vanishing of bounded cohomology with coefficients in a certain class of dual modules. Such characterizations were obtained in [40] and [171]. A characterization of property A for metric spaces in terms of a different cohomology theory was given in [43]. A homological characterization of property A was proved in [41].

Recently several new characterizations and connections between property A and other large scale geometric properties were found. In [42] it was shown that property A implies the operator norm localization property introduced in [57]. In [217] it was proved that these two notions are in fact equivalent for bounded geometry metric space.

Chapter 5
Coarse embeddings

A coarse embedding is a notion of inclusion suitable for large scale geometry. It has received significant attention after the results in [250], where certain topological conjectures were proved under the assumption of existence of such coarse embeddings into Hilbert spaces.

5.1 Coarse embeddings

Coarse embeddings were introduced by Gromov [108], as a natural notion of inclusion, up to coarse equivalence, in large scale geometry. We have already defined coarse embeddings in the first chapter. Below we recall the definition.

Definition 5.1.1. Let X and Y be metric spaces. A map $f\colon X \to Y$ is called a *coarse embedding* if there exist two non-decreasing functions $\rho_-, \rho_+\colon [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_-(t) = \infty$ and

$$\rho_-(d_X(x,y)) \le d_Y(f(x), f(y)) \le \rho_+(d_X(x,y))$$

for every $x, y \in X$.

In other words, a coarse embedding of X into Y is a coarse equivalence of X with a subset of Y. Note that a coarse embedding is in fact a quasi-isometric embedding whenever ρ_- and ρ_+ are affine functions.

The intuition behind coarse embeddability is that we are trying to draw a "nice" picture of one metric space in another one, so that this picture reflects the large scale geometry of the embedded space. It is not always possible to do this isometrically (i.e., preserving the metric exactly), or even bi-Lipschitzly (i.e., preserving the metric up to a fixed multiplicative constant) , but we will only care about large distances being preserved uniformly.

Example 5.1.2. Any isometric embedding is a coarse embedding. In particular, a metric space X admits an isometric, and in thus coarse, embedding into the Banach space $\ell_\infty(X)$. To see this, fix $x_0 \in X$ and define the *Kuratowski embedding* $F\colon X \to \ell_\infty(X)$ by setting

$$F(x) = f^{(x)},$$

where

$$f^{(x)}(y) = d(x,y) - d(y,x_0)$$

for all $x, y \in X$. The function $f^{(x)}$ is indeed bounded for every x since, by the triangle inequality, we have $d(x,z) - d(z,x_0) \leq d(x,x_0)$. The distance between the images of x and y satisfies

$$\sup_{z \in X} |f^{(x)}(z) - f^{(y)}(z)| = \sup_{z \in X} |d(x,z) - d(y,z)| \leq d(x,y).$$

On the other hand, taking $z = x$ or $z = y$ we bound the first expression from below by $d(x,y)$.

5.2 Embeddability into Hilbert spaces

We will be mainly interested in coarse embeddings of metric spaces into various Banach spaces, such as Hilbert spaces.

Example 5.2.1. Equip $\mathbb{R}^n$ with the ℓ_1-metric. Clearly, $\mathbb{Z}^n$ embeds coarsely into the linear space $\mathbb{R}^m$, with the ℓ_1-metric, whenever $m \geq n$. The ℓ_1-metric on $\mathbb{R}^m$ is bi-Lipschitz equivalent to the usual Euclidean metric since they are both induced by norms and any two norms on a finite-dimensional vector space are equivalent. Thus $\mathbb{Z}^n$ embeds coarsely into the m-dimensional Euclidean space for $m \geq n$.

In the above example the embedding is in fact bi-Lipschitz. The next example is not a bi-Lipschitz embedding but does preserve large distances uniformly.

Example 5.2.2. Let $T = (V, E)$ be a tree in which every vertex has finite degree. Fix a vertex $x_0 \in T$ and for every point $x \in V$ consider a chain $x_0, x_1, \dots, x_n = x$ of vertices on the unique geodesic segment connecting x_0 and x: $d(x_i, x_{i+1}) = 1$ and $d(x_0, x) = n$. Construct a map $F \colon V \to \ell_2(V)$ by setting

$$F(x) = \sum_{i=0}^{d(x_0,x)} 1_{x_i}$$

for every $x \in V$. Intuitively, we are mapping each new vertex of the tree into a new orthogonal direction in the Hilbert space. Given $x, y \in T$ consider the vertices t_i, lying on the unique geodesic segment connecting x and y: $x = t_0$, $y = t_n$, $d(t_i, t_{i+1}) = 1$, where $n = d(x, y)$. Then it is easy to verify that

$$\|F(x) - F(y)\|_2 = \Big(\sum_{i=1}^{d(x,y)} 1_{t_i} \Big)^{1/2} = d(x,y)^{1/2}$$

for any $x, y \in V$. It follows that F is a coarse embedding with $\rho_-(t) = \rho_+(t) = t^{1/2}$.

Example 5.2.3. This example was given in [209]. The Banach space $\ell_1(\mathbb{N})$ embeds coarsely into a Hilbert space. Put

$$H = \Big(\bigoplus_{n\in\mathbb{N}} L_2(\mathbb{R})\Big)_{(2)}$$

and let $f : \ell_1(\mathbb{N}) \to H$ be defined as

$$f((a_1, a_2, a_3, \dots)) = \bigoplus_{i=1}^{\infty} \zeta(a_i),$$

where $\zeta(a_i) \colon \mathbb{R} \to \mathbb{R}$,

$$\zeta(a_i) = \begin{cases} 1_{[0,a_i]} & \text{if } a_i \geq 0, \\ 1_{[a_i,0]} & \text{if } a_i < 0. \end{cases}$$

Here $(a_1, a_2, a_3, \dots)$ is an element of $\ell_1(\mathbb{N})$. One verifies that

$$\|f(a) - f(a')\|_H = \sqrt{\|a - a'\|_{\ell_1}}$$

for any $a, a' \in \ell_1(\mathbb{N})$, so f is a coarse embedding with $\rho_-(t) = \rho_+(t) = t^{1/2}$.

A natural question is how to construct coarse embeddings for other examples of metric spaces. For uniformly discrete metric spaces the following statement provides a useful sufficient condition.

Theorem 5.2.4. *Let X be a metric space. If X has property A then it embeds coarsely into the Hilbert space.*

In the previous chapter we established several techniques to verify property A for various classes of groups. The above theorem will now allow us to use these facts to construct coarse embeddings into a Hilbert space. In fact, the above theorem was the original motivation for introducing property A.

In order to prove the theorem we first establish the following helpful fact.

Lemma 5.2.5. *Let X be a metric space with property A. Then for every $R > 0$ and $\varepsilon > 0$ there exists a map $\xi \colon X \to \ell_2(X \times \mathbb{N})$ and $S > 0$ such that*

(1) $\|\xi_x\|_2 = 1$,
(2) $\|\xi_x - \xi_y\|_2 \leq \varepsilon$ *if* $d(x, y) \leq R$, *and*
(3) $\operatorname{supp} \xi_x \subseteq B(x, S) \times \mathbb{N}$

for every $x, y \in X$.

Proof. Consider the space $\ell_2(X \times \mathbb{N})$ and denote by $\|\cdot\|_2$ the norm in this Hilbert space. For $\varepsilon > 0$ and $R > 0$ consider a family of sets $\{A_x\}_{x\in X}$ given by the definition of property A for ε and R. We construct $\xi_x \in \ell_2(X \times \mathbb{N})$ by setting

$$\xi_x = \frac{1_{A_x}}{\sqrt{\#A_x}}$$

for every $x \in X$. Then, clearly, the first and last conditions are satisfied and we only need to prove the ε-condition holds.

The ε-condition in the definition of property A for ε', to be chosen later, and $R > 0$ yields

$$\#A_x + \#A_y = 2\#(A_x \cap A_y) + \#(A_x \mathbin{\triangle} A_y) \leq (2+\varepsilon')\#(A_x \cap A_y)$$

whenever $d(x,y) \leq R$. This implies that

$$\langle \xi_x, \xi_y \rangle = \frac{\#(A_x \cap A_y)}{\sqrt{(\#A_x)(\#A_y)}} \geq \frac{2\#(A_x \cap A_y)}{\#A_x + \#A_y} \geq \frac{2}{2+\varepsilon'}$$

whenever $d(x,y) \leq R$. It follows that

$$\|\xi_x - \xi_y\|_2^2 \leq \frac{2\varepsilon'}{2+\varepsilon'} \leq \varepsilon'.$$

It suffices to take $\varepsilon' = \varepsilon^2$ to see that the required conditions are satisfied. $\square$

Proof of Theorem 5.2.4. For every $n \in \mathbb{N}$ choose ξ^n as in Lemma 5.2.5, with $\varepsilon = 2^{-n}$ and $R = n$. We choose a sequence of positive numbers S_n such that $\operatorname{supp} \xi_x^n \subseteq B(x, S_n)$, which implies

$$\|\xi_x^n - \xi_y^n\|_2 = \sqrt{2}$$

when $d(x,y) \geq 2S_n$. Without loss of generality we can assume that the sequence $\{S_n\}$ is increasing.

Fix $z \in X$ and define a map $F\colon X \to \bigoplus_{n=1}^{\infty} \ell_2(X \times \mathbb{N})$ by the formula

$$F(x) = \bigoplus_{n=1}^{\infty} (\xi_x^n - \xi_z^n).$$

We need to show that F is well defined and that it is a coarse embedding. Observe that if k is such that $k - 1 < d(x,y) \leq k$, then

$$\|\xi_x^n - \xi_y^n\|_2 \leq \frac{1}{2^n}$$

for $n \geq k$. Hence,

$$\begin{aligned}\|F(x) - F(y)\|^2 &= \sum_{n=1}^{\infty} \|\xi_x^n - \xi_y^n\|_2^2 \\ &\leq \sum_{n=1}^{k} \|\xi_x^n - \xi_y^n\|_2^2 + \sum_{n=k+1}^{\infty} \frac{1}{2^n} \\ &\leq 2k + 1 \leq 2d(x, y) + 3.\end{aligned}$$

First, note that if $y = z$, then the above computation shows that each $F(x)$ is a well-defined vector in $\bigoplus \ell_2(X \times \mathbb{N})$. The second consequence is that we obtain an upper bound

$$\rho_+(t) = (2t + 3)^{1/2}$$

since

$$\|F(x) - F(y)\| \leq (2d(x, y) + 3)^{1/2}.$$

On the other hand, if we let $\phi(k) = \sup\{n \in \mathbb{N} \mid 2S_n \leq k - 1\}$, then

$$\|F(x)-F(y)\| = \sum_{n=1}^{\phi(k)} \|\xi_x^n-\xi_y^n\|_2 + \sum_{n=\phi(k)+1}^{\infty} \|\xi_x^n-\xi_y^n\|_2 \geq \sum_{n=1}^{\phi(k)} \|\xi_x^n-\xi_y^n\|_2 = 2\phi(k).$$

Since S_n is an increasing sequence, the function $\phi(k)$ is a non-decreasing function of k, with the property that $\phi(k) \to \infty$ as $k \to \infty$. Thus we have

$$\sqrt{\overline{\phi}(d(x, y))} \leq \|F(x) - F(y)\|,$$

where $\overline{\phi}$ is a piecewise linear extension of ϕ to the positive reals. □

The above construction will be used extensively in this and the following chapters. Its origins can be traced back to an article of Bekka, Cherix and Valette [27], where this construction was used to answer a question of Gromov whether amenable groups admit a metrically proper, affine, isometric actions on Hilbert spaces. We will discuss such actions in Chapter 6.

As a consequence of the above theorem we have the following

Corollary 5.2.6. *Let X be a uniformly discrete metric space with finite asymptotic dimension. Then X admits a coarse embedding into a Hilbert space.*

Proof. Finite asymptotic dimension implies property A, by Theorem 4.3.6, and the existence of the embedding follows from Theorem 5.2.4. □

Applying Theorem 4.3.8 we also obtain

Corollary 5.2.7. *Let X be a uniformly discrete metric space with finite decomposition complexity. Then X embeds coarsely into a Hilbert space.*

Using our previous observation, relating asymptotic dimension and property A, we conclude that the following spaces embed coarsely into a Hilbert space:

- amenable groups,
- almost connected Lie groups,
- hyperbolic groups,
- linear groups.

Coarse embeddings of hyperbolic groups into a Hilbert space were constructed originally by Sela [222]. In [113] it was proved that linear groups have property A.

The construction used in the proof of Theorem 5.2.4 can be used to prove a more general fact, a characterization of coarse embeddability into a Hilbert space.

Theorem 5.2.8. *A uniformly discrete metric space X admits a coarse embedding into a Hilbert space if and only if for every $\varepsilon > 0$ and every $R > 0$ there exists a Hilbert space H, a map $\xi\colon X \to H$ and a non-decreasing function $\varphi\colon [0,\infty) \to (0,\sqrt{2})$, $\lim_{t\to\infty} \varphi(t) = \sqrt{2}$, satisfying the following conditions:*

(1) $\|\xi_x\| = 1$,

(2) $\|\xi_x - \xi_y\| \le \varepsilon$ *if* $d(x,y) \le R$, *and*

(3) $\|\xi_x - \xi_y\| \ge \varphi(d(x,y))$

for every $x, y \in X$.

Proof. The "if" direction can be proved using the construction in the proof of Theorem 5.2.4. The details are left to the reader as an exercise.

To prove the other direction assume that we have a coarse embedding $F\colon X \to H$, into a separable, real Hilbert space H. We construct a new, real Hilbert space,

$$\operatorname{Exp}(H) = \mathbb{R} \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus \cdots = \Big(\bigoplus_{n=0}^{\infty} H^{\otimes n}\Big)_{(2)}.$$

This new space[1] allows us to exponentiate vectors in the following way. Define the exponential map $\exp\colon H \to \operatorname{Exp}(H)$, by setting

$$\exp(v) = 1 \oplus v \oplus \left(\frac{1}{\sqrt{2!}}\, v \otimes v\right) \oplus \left(\frac{1}{\sqrt{3!}}\, v \otimes v \otimes v\right) \oplus \cdots$$

(note that we are using here the definition of the exponential function in terms of infinite series). Then we have

$$\langle \exp(v), \exp(w)\rangle = e^{\langle v,w\rangle}$$

for any $v, w \in H$.

[1] This space is also known as *Fock space*.

Let $t > 0$ and define

$$\xi_x = e^{-t\|F(x)\|^2} \exp(\sqrt{2t}F(x))$$

for every $x \in X$. Then

$$\langle \xi_x, \xi_y \rangle = \exp(-t\|F(x) - F(y)\|^2),$$

and $\|\xi_x\| = 1$ for every $x \in X$. Also,

$$e^{-t\rho_+(d(x,y))^2} \le \langle \xi_x, \xi_y \rangle \le e^{-t\rho_-(d(x,y))^2}$$

for all $x, y \in X$. To obtain the required conditions for R and ε, we choose

$$t = \frac{\varepsilon}{1 + \rho_+(R)^2},$$

and the estimate follows from the parallelogram identity. □

Example 5.2.9. Consider the construction of property A for a tree T described in Example 4.1.5. For $n \in \mathbb{N}$, $\varepsilon = 2/n$ and $R > 0$ one takes A_x to be the vertices on the geodesic segments of length Rn, originating at x, in the direction of a fixed geodesic ray ω.

Define $\xi \colon T \to \ell_2(T)$ by setting

$$\xi_x = \frac{1_{A_x}}{\|1_{A_x}\|_2}.$$

Note that for any $x \in T$ we have $\|1_{A_x}\|_2 = \sqrt{nR}$ and

$$\xi_x = \frac{1_{A_x}}{\sqrt{nR}}.$$

Then

$$\|\xi_x - \xi_y\|_2 = \frac{\|1_{A_x} - 1_{A_y}\|_2}{\sqrt{nR}} = \frac{\sqrt{\#(A_x \triangle A_y)}}{\sqrt{nR}} = \frac{\sqrt{2R}}{\sqrt{nR}}$$

whenever $d(x, y) \le R$. On the other hand, ξ_x and ξ_y are orthogonal when $d(x, y) \ge 2Rn$, which implies that $\|\xi_x - \xi_y\|_2 = \sqrt{2}$. Given $\varepsilon > 0$ and $R > 0$ we obtain the required condition by choosing appropriately large $n \in \mathbb{N}$.

5.3 Examples of embeddable spaces without property A

In this section we will give examples showing that Theorem 5.2.4 does not have a converse in the class of uniformly discrete metric spaces.

Locally finite examples

We first return to the example discussed in Theorem 4.5.3, the disjoint union of Cartesian powers of a finite, nontrivial group. In Theorem 4.5.3 we have proved that this space does not have property A.

Theorem 5.3.1. *Let G be a finite, non-trivial group. Then the coarse disjoint union of $\{G^n\}_{n\in\mathbb{N}}$, considered in Theorem 4.5.3, embeds coarsely into a Hilbert space.*

Proof. Since the space is a disjoint union of finite spaces, it is enough to show that all the G^n can be embedded with the same functions ρ_-, ρ_+ into a Hilbert space (see Exercise 5.1). Take any injective function

$$f\colon G\to \ell_1(\mathbb{N}).$$

Since G is finite, there exists a constant $L>0$ such that

$$\frac{1}{L}d_G(g,h)\le \|f(g)-f(h)\|_1\le L d_G(g,h).$$

It is easy to check that for every $n\in\mathbb{N}$ the map

$$f^n=\underbrace{f\times\cdots\times f}_{n\text{ times}}\colon G^n\to\Big(\bigoplus_{i=1}^{n}\ell_1\Big)_{(1)},$$

also satisfies

$$\frac{1}{L}d_{G^n}(g,h)\le \|f^n(g)-f^n(h)\|_1\le L d_{G^n}(g,h)$$

for every $g,h\in G^n$. Since $(\bigoplus_{i=1}^n \ell_1(\mathbb{N}))_{(1)}$ is isometrically isomorphic to $\ell_1(\mathbb{N})$, the map f^n can be viewed as a map $f^n\colon G^n\to\ell_1(\mathbb{N})$, and this proves that the disjoint union $\coprod_{n=1}^\infty G^n$ embeds bi-Lipschitzly into $\ell_1(\mathbb{N})$. By Example 5.2.3, $\ell_1(\mathbb{N})$ embeds coarsely into a Hilbert space and the composition of these two embeddings gives a coarse embedding of $\coprod G^n$ into a Hilbert space. □

Bounded geometry examples

Examples with bounded geometry were constructed by Arzhantseva, Guentner and Špakula [10]. Consider a discrete group G and denote by $G^{(2)}$ the normal subgroup generated by all elements of the form g^2 for $g\in G$. Define a sequence of subgroups of the free group $\mathbb{F}_2$ as follows:

$$G_0=\mathbb{F}_2,\quad G_n=G_{n-1}^{(2)}\quad \text{for } n=1,2,\dots.$$

The sequence of subgroups $\{G_n\}$ satisfies $\bigcap_{n=1}^\infty G_n=\{e\}$. This fact is not immediate, it can be proved using Levi's theorem (see [154]). Define the space (X,d_B) to be the coarse disjoint union of finite groups $\{\mathbb{F}_2/G_n\}_{n\in\mathbb{N}}$.

Theorem 5.3.2 ([10]). *The space X does not have property A and embeds coarsely into a Hilbert space.*

Sketch of proof. By Theorem 4.4.6, the space X does not have property A and the main difficulty is to show that the above space embeds coarsely into a Hilbert space.

The proof relies on the fact that the finite quotients G/G_n can be given a structure of a *wall space*. This structure is present thanks to the fact that the sequence of finite quotients is a tower of graph coverings, called $\mathbb{Z}_2$*-homology covers*. A wall is partition of a set X into a subset W and its complement W^c. A wall structure on a set is then a collection of walls such that the number of walls separating any two points is positive and finite. Such a wall structure gives rise to a metric by defining the distance between two points to be the number of walls separating them. Such metrics are well known to embed isometrically into a certain L_1-space, namely the space $L_1(W)$, where W is the set of walls. More precisely, one defines the embedding to be $f(x) = 1_{W(x|x_0)}$, where $W(x|x_0)$ is the set of walls separating x and a distinguished point x_0. Since L_1 embeds coarsely into a Hilbert space, the metric space X also embeds coarsely into a Hilbert space.

The wall structure on the coarse disjoint union X above thus gives rise to a metric d_W and the metric space (X, d_W) is coarsely embeddable into the Hilbert space. The metrics d_W and d_B do not agree. However, it turns out that d_W and d_B are coarsely equivalent, which allows one to conclude that X, with the metric d_B, admits a coarse embedding into a Hilbert space. □

A similar construction, inspired by the one above, was given subsequently by Ostrovskii [193]. Starting with a family of graphs $\{G_i\}$ whose girths tend to infinity, the graph X_i is constructed as a $\mathbb{Z}_2$-homology cover of G_i. The X_i have the property that each one admits an L-bi-Lipschitz embedding into the Banach space L_1 for a uniform constant L. In addition, the girths of the X_i's tend to infinity with i. By Theorem 4.4.9, this last fact implies that the coarse disjoint union of X_i does not have property A. A generalization of the construction used in [10] was also studied in [146].

In view of the above examples the following question remains open.

Open question 5.3.3. *Does there exist a finitely generated group which embeds coarsely into a Hilbert space but does not have property A?*

Conditions under which coarse embeddability implies property A will be discussed in the following sections.

5.4 Convexity and reflexivity

In infinite-dimensional Banach spaces one often quantifies how close the geometry of the given space is to the geometry of the Hilbert space. This can be done using convexity notions such as uniform convexity, uniform smoothness, type and cotype. We will discuss embeddings into Banach spaces, possessing various convexity conditions. The weakest of such properties is strict convexity.

Definition 5.4.1. A Banach space E is called *strictly convex* if for every pair of vectors $v, w \in E$ such that $v \neq w$ and $\|v\|_E = \|w\|_E = 1$, we have $\|\frac{v+w}{2}\|_E < 1$.

In other words, for two vectors in the unit sphere, their middle point is not on the sphere. The question of how restrictive is the property of embeddability into strictly convex Banach spaces has an immediate answer.

Theorem 5.4.2. *Every countable discrete metric space admits a bi-Lipschitz, and thus coarse, embedding into a strictly convex Banach space.*

We will only sketch the proof. We refer to [1], [31], [162] for background on Banach spaces. It is a well-known fact that a separable metric space embeds isometrically into $C(X)$, where the latter is the linear space of continuous functions $f: X \to \mathbb{R}$, on a second countable compact Hausdorff space X, equipped with the norm

$$\|f\|_{C(X)} = \sup_{x \in X} |f(x)|.$$

$C(X)$ itself is not strictly convex, however, it is separable, and it is a well-known fact in Banach space theory that any separable Banach space can be equipped with an equivalent norm which is strictly convex [139].

Recall that a Banach space E is reflexive if the natural inclusion $E \subseteq E^{**}$ into the double dual of E is an isomorphism. Reflexivity is a more restrictive condition than that of strict convexity, as there are separable Banach spaces, such as c_0 or ℓ_1, which are not reflexive. A natural question is which metric spaces admit embeddings into reflexive Banach spaces. The next theorem, due to Brown and Guentner [45], shows that this requirement is also not restrictive, at least if we restrict our attention to bounded geometry metric spaces.

Theorem 5.4.3. *Let X be a bounded geometry metric space. Then X admits a coarse embedding into a reflexive Banach space. More precisely, there exists an increasing sequence of numbers p_n, satisfying $\lim_{n\to\infty} p_n = \infty$, such that X embeds into the ℓ_2-direct sum $(\bigoplus_{n=1}^{\infty} \ell_{p_n}(X))_{(2)}$.*

For the proof we need a lemma which states that every uniformly discrete metric space has a version of property A with respect to the ℓ_∞ norm.

Lemma 5.4.4 ([45], [69]). *Let X be a uniformly discrete metric space. Then for every $n \in \mathbb{N}$ there exists a map $\xi^n : X \to \ell_\infty(X)$ such that*

(1) $\|\xi^n_x\|_\infty = 1$,

(2) $\|\xi^n_x - \xi^n_y\|_\infty \leq \dfrac{d(x, y)}{n}$, *and*

(3) $\operatorname{supp} \xi_x^n \subseteq B(x,n)$

for every $x, y \in X$.

Proof. For $n \in \mathbb{N}$ define

$$\xi_x^n(y) = \max\{1 - \tfrac{1}{n} d(x,y), 0\}.$$

The first condition is obvious. Clearly, $\operatorname{supp} \xi_x^n \subseteq B(x,n)$ for every $x \in X$, so the last condition holds as well. To prove the second condition we note that the expression $|\xi_x^n(z) - \xi_y^n(z)|$ has the following possible values:

$$|\xi_x^n(z) - \xi_y^n(z)| = \begin{cases} \frac{1}{n}|d(x,z) - d(y,z)| & \text{if } d(x,z) \le n \text{ and } d(y,z) \le n, \\ 1 - \frac{d(x,z)}{n} & \text{if } n < d(y,z) \text{ and } d(x,z) \le n, \\ 1 - \frac{d(y,z)}{n} & \text{if } n < d(x,z) \text{ and } d(y,z) \le n, \\ 0 & \text{if } d(x,z) > n \text{ and } d(y,z) > n. \end{cases}$$

The first value can be estimated as

$$\frac{|d(x,z) - d(y,z)|}{n} \le \frac{d(x,y)}{n}.$$

For the second expression we have

$$1 < \frac{d(y,z)}{n} \le \frac{d(x,y) + d(x,z)}{n},$$

which gives

$$1 - \frac{d(x,z)}{n} \le \frac{d(x,y)}{n},$$

as required. The estimate for the third expression is similar. Then the norm $\|\xi_x^n - \xi_y^n\|_\infty$ is the largest of the above four values, all of which are at most $d(x,y)/n$. □

The proof of Theorem 5.4.3 is based on the idea that in finite-dimensional spaces the ℓ_p-norms approximate the ℓ_∞-norm as p increases to infinity. More precisely, for any finitely supported $f : X \to \mathbb{R}$ we have

$$\|f\|_\infty \le \Big(\sum_{x \in X} |f(x)|^p\Big)^{1/p} \le \|f\|_\infty \Big(\sum_{x \in \operatorname{supp} f} 1\Big)^{1/p} = \|f\|_\infty (\# \operatorname{supp} f)^{1/p}.$$

Proof of Theorem 5.4.3. Let X be a metric space with bounded geometry. For every $n \in \mathbb{N}$ we have $\# B(x,n) \le C(n)$, where $C(n) > 0$ is a constant, which depends only on n. Thus, for every $n \in \mathbb{N}$ and $\xi^n : X \to \ell_\infty(X)$, as in Lemma 5.4.4, we can find sufficiently large $p_n \ge 1$ such that

$$(\# \operatorname{supp}(\xi_x^n - \xi_y^n))^{1/p_n} \le 1 + \frac{1}{n},$$

since $\#\operatorname{supp}(\xi_x^n - \xi_y^n)$ has an upper bound depending on n, but not on x and y. Then

$$\begin{aligned}\|\xi_x^n - \xi_y^n\|_{p_n} &\leq (\#\operatorname{supp}(\xi_x^n - \xi_y^n))^{1/p_n}\|\xi_x^n - \xi_y^n\|_\infty \\ &\leq \|\xi_x^n - \xi_y^n\|_\infty \Big(1 + \frac{1}{n}\Big) \\ &\leq \frac{2d(x,y)}{n}.\end{aligned}$$

Fix $x_0 \in X$ and define a map

$$F\colon X \longrightarrow \Big(\bigoplus_{n=1}^{\infty} \ell_{p_n}(X)\Big)_{(2)},$$

where

$$F(x) = \bigoplus_{n=1}^{\infty} (\xi_x^n - \xi_{x_0}^n)$$

for every $x \in X$. Note that for any $y \in X$ we have

$$\|F(x) - F(y)\|^2 = \sum_{n=1}^{\infty} \|\xi_x^n - \xi_y^n\|_{p_n}^2 \leq (2d(x,y))^2 \Big(\sum_{n=1}^{\infty} \frac{1}{n^2}\Big).$$

The verification that F is a coarse embedding proceeds as in the proof of Theorem 5.2.8. □

Theorem 5.4.3 can be strengthened. Cotype is a measure of convexity of a Banach space. A Banach space X is said to have trivial cotype if for every $\varepsilon > 0$ the finite-dimensional spaces ℓ_∞^n are isomorphic to subspaces of X, via isomorphisms T_n satisfying $\|T_n\|\,\|T_n^{-1}\| \leq (1+\varepsilon)$ independently of n. It is shown in [21] that any locally finite metric space embeds bi-Lipschitzly into any Banach space with trivial cotype. The same statement for coarse embeddings was shown independently in [191].

The above result shows that reflexivity is not strong enough to give an interesting restriction on the property of coarse embeddability, and we will introduce a notion of convexity that is stronger than the ones discussed so far.

Definition 5.4.5. A Banach space E is called *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any vectors $v, w \in E$ satisfying $\|v\|_E = 1$ and $\|w\|_E = 1$ the condition $\|v - w\|_E \geq \varepsilon$ implies $\|\frac{v+w}{2}\|_E \leq 1 - \delta$.

Uniform convexity thus states that for unit vectors at uniform distance their midpoint is uniformly bounded away from the unit sphere. Uniform convexity was introduced by Clarkson [58]. Examples of uniformly convex spaces include the spaces ℓ_p for all $1 < p < \infty$, this can be proved using the so-called Clarkson's inequalities. On the other hand, the spaces ℓ_1 and ℓ_∞ are not uniformly convex, which can be seen already in two dimensions. We leave this last fact as an exercise.

We will now prove that the Hilbert space is the most difficult to embed into among all the spaces ℓ_p. For this we will need the Mazur map. Given a set X and $1 \leq p, q < \infty$ we define the *Mazur map* $M_{p,q} \colon \ell_p(X) \to \ell_q(X)$ by the formula

$$M_{p,q}(f)(x) = |f(x)|^{p/q} \operatorname{sign}(f(x))$$

for every $f \in \ell_p(X)$. It is not hard to see that

$$\|M_{p,q}(f)\|_q = \|f\|_p$$

for any $f \in \ell_p(X)$. We also note that $M_{p,q} \circ M_{q,p}$ is the identity on $\ell_p(X)$, so that $M_{p,q}$ and $M_{q,p}$ are inverses of each other. An important property of the Mazur map is, that as a map between unit spheres, it is a uniformly continuous homeomorphism with a uniformly continuous inverse.

Proposition 5.4.6. *Let $1 \leq p < q < \infty$. Then there exists a constant $C > 0$, which depends only on p and q, such that*

$$\frac{p}{q}\|f - g\|_p \leq \|M_{p,q}(f) - M_{p,q}(g)\|_q \leq C\,\|f - g\|_p^{p/q}$$

for every $f, g \in \ell_p(X)$ satisfying $\|f\|_p = \|g\|_p = 1$.

We refer to [31] for the proof. Note that the above proposition also covers the case $p > q$ by representing f and g as images of elements of $\ell_q(X)$ under $M_{q,p}$. Uniformly continuous maps are a classical topic in the geometry of Banach spaces; see [31] for details.

Theorem 5.4.7 ([181]). *Let X be a metric space which embeds coarsely into a Hilbert space. Then X embeds coarsely into ℓ_p for any $p \geq 1$.*

Proof. Since X embeds coarsely into the Hilbert space, by Theorem 5.2.8 for every $n \in \mathbb{N}$ there exists a function $\xi^n \colon X \to \ell_2(\mathbb{N})$ and a constant $S_n > 0$ such that $\|\xi^n_x\| = 1$, $\|\xi^n_x - \xi^n_y\| \leq 2^{-n}$ when $d(x, y) \leq n$ and $\|\xi^n_x - \xi^n_y\| \geq 1$ whenever $d(x, y) \geq S_n$. Let $p \geq 2$ and define $\eta^n \colon X \to \ell_p(\mathbb{N})$ by setting

$$\eta^n_x = M_{2,p}(\xi^n_x).$$

We then have $\|\eta^n_x\|_p = 1$ for every $x \in X$ and $n \in \mathbb{N}$. Also, by Proposition 5.4.6, we have

$$\|\eta^n_x - \eta^n_y\|_p \leq C\frac{1}{2^{2n/p}},$$

whenever $d(x, y) \leq n$ and

$$\|\eta^n_x - \eta^n_y\| \geq \frac{2}{p}$$

when $d(x, y) \geq S_n$.

We now construct a coarse embedding in the usual way. Fix a point $x_0 \in X$ and define a map $F\colon X \to (\bigoplus_{n=1}^{\infty} \ell_p(\mathbb{N}))_{(p)}$ by setting

$$F(x) = \bigoplus_{n=1}^{\infty} (\eta_x^n - \eta_{x_0}^n)$$

for every $x \in X$. The verification that F is a coarse embedding is similar to the previous cases. The proof of the case $p \leq 2$ is left to the reader. $\square$

In fact it is true that the properties of coarse embeddability into ℓ_p for $1 \leq p \leq 2$ are all equivalent [181]. Theorem 5.4.7 can be generalized to a larger class of Banach spaces by using a generalized version of the Mazur map, due to Odell and Schlumprecht [188], which is a uniform homeomorphism between the unit spheres of the Hilbert space and any *uniformly convex Banach space with unconditional basis*. When $p \geq 2$ the situation is very different. For $p \geq q \geq 2$ the space ℓ_q embeds coarsely into ℓ_p [163], Remark 5.10, but ℓ_p does not embed coarsely into ℓ_q [164]. See [31] for background on Banach space theory.

5.5 Coarse embeddings and finite subsets

It is clear that if a bounded geometry metric space X embeds coarsely into the Hilbert space with bounds ρ_- and ρ_+, then the restriction of the embedding to any subset $A \subseteq X$ is also a coarse embedding, with the same bounds ρ_- and ρ_+. The main goal of this section is to prove the converse statement obtained in [75].

Theorem 5.5.1. *Let X be a metric space. If there exist non-decreasing functions $\rho_-, \rho_+\colon [0,\infty) \to [0,\infty)$, $\lim \rho_-(t) = \infty$ such that for every finite subset $A \subseteq X$ there exists a map $f_A\colon A \to H$ to a Hilbert space satisfying*

$$\rho_-(d(x,y)) \leq \|f_A(x) - f_A(y)\|_H \leq \rho_+(d(x,y))$$

for every $x, y \in A$, then X admits a coarse embedding into a Hilbert space with bounds ρ_- and ρ_+.

To prove this theorem we will need a new notion.

Definition 5.5.2. Let X be a set. A function $K\colon X \times X \to \mathbb{R}$ is a *positive definite kernel* on X if for any choice of finitely many points $x_1, \dots, x_n \in X$ and any choice of numbers $a_1, \dots, a_n \in \mathbb{R}$, we have

$$\sum_{i,j} a_i a_j K(x_i, x_j) \geq 0.$$

Equivalently, K is positive definite if for any finite subset $F \subseteq X$ the finite matrix K_F, obtained by restricting K to F, is positive definite; that is

$$\langle K_F a, a\rangle \geq 0$$

for any vector $a \in \mathbb{R}$.

Example 5.5.3. Let X be a set, H be a Hilbert space and let $\xi\colon X \to H$ be any map. Then the kernel

$$K(x, y) = \langle \xi_x, \xi_y\rangle_H,$$

where $x, y \in X$, is a positive definite kernel on X. Indeed, for any finite set $F \subseteq X$ we have

$$\sum_{x,y\in F} \langle \xi_x, \xi_y\rangle a_x a_y = \Big\| \sum_{x\in F} a_x \xi_x \Big\|^2 \geq 0.$$

It turns out that the above example is the only one possible.

Theorem 5.5.4. *Let K be a positive definite kernel on a set X. Then there exists a Hilbert space H and a map $\xi\colon X \to H$ such that*

$$K(x, y) = \langle \xi_x, \xi_y\rangle_H.$$

Proof. Consider the vector space $C_c(X, \mathbb{R})$ of finitely supported functions $f\colon X \to \mathbb{R}$. The formula

$$\langle f, f'\rangle = \sum_{x,y\in X} K(x, y) f(x) f'(y),$$

for $f, f' \in C_c(X, \mathbb{R})$, defines an inner product on $C_c(X, \mathbb{R})$ since K is positive definite. It might happen that $\langle f, f\rangle_K = 0$. By taking $N = \{f \mid \langle f, f\rangle_K = 0\}$ we can define H to be the completion of $C_c(X, \mathbb{R})/N$ under the norm induced by the inner product. Then the map $\xi\colon X \to H$ defined by $\xi_x = 1_x$ satisfies the necessary conditions. □

Proof of Theorem 5.5.1. Fix $0 < t < 1$. Applying the construction in the proof of Theorem 5.2.8 to every f_A, we conclude that for every finite subset $A \subseteq X$ there exists a map $\xi^A\colon A \to H$ such that $\|\xi_x^A\| = 1$ for every $x \in A$ and

$$e^{-t\rho_+(d(x,y))^2} \leq \langle \xi_x^A, \xi_y^A\rangle \leq e^{-t\rho_-(d(x,y))^2}$$

for every $x, y \in A$. Defining

$$K_A(x, y) = \begin{cases} \langle \xi_x, \xi_y\rangle & \text{for } x, y \in A, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain a positive definite kernel (Exercise 5.7). The K_A form a net, directed by inclusion of the indexing subsets. Let K denote the limit of a convergent subnet in the topology of pointwise convergence (such a convergent subnet exists by compactness). Then K is a positive definite kernel satisfying

$$e^{-t\rho_+(d(x,y))^2} \leq K(x,y) \leq e^{-t\rho_-(d(x,y))^2}$$

for every $x, y \in X$ (see Exercise 5.8). Theorem 5.5.4 and a choice of t similar to the proof of Theorem 5.2.8 now gives $\xi\colon X \to H$ as in Theorem 5.2.8, and the claim follows. ☐

5.6 Expanders

Expanders are finite graphs which are *highly connected* but *sparse* at the same time. They have many remarkable properties; for us they will serve as examples of spaces which do not embed coarsely into a Hilbert space. Recall that a graph is called k-regular if every vertex has degree exactly k. A family of graphs is said to have bounded degree if there is a constant $K > 0$ such that the degree of every vertex in every graph in the family is at most K. All graphs considered here are assumed to be connected, in the sense that any two vertices in a single graph are joined by a path of edges.

Definition 5.6.1. Let $X = (V, E)$ be a finite k-regular graph. Define

$$\lambda_1(X) = \inf \frac{\sum_{x\in V}\sum_{y\in B(x,1)} |f(x) - f(y)|^2}{\sum_{x\in V} |f(x)|^2}, \tag{5.1}$$

where the infimum is taken over all non-zero functions $f : V \to \mathbb{R}$ satisfying

$$\sum_{x\in V} f(x) = 0.$$

Note that the condition $y \in B(x,1)$, $y \neq x$, is equivalent to the condition x and y are connected by an edge. Consider a finite connected graph $X = (V, E)$ and a linear operator $\nabla\colon \ell_2(V) \to \ell_2(E_\pm)$, where $E_\pm$ denotes the set of oriented edges of X, with some prescribed orientation given by the formula

$$(\nabla f)(x,y) = \begin{cases} f(y) - f(x) & \text{if } d(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

The operator ∇ is the *discrete gradient* and, clearly, $\nabla f = 0$ if and only if f is constant on V. Note that $\ell_2(V)$ can be represented as a direct sum of $\ell_2^0(V) =$

$\{f : V \to \mathbb{R} \mid \sum_{x \in V} f(x) = 0\}$ and $\mathbb{C} = \{f : G \to \mathbb{R} \mid f \equiv \text{const}\}$. The orthogonal projections onto these subspaces are

$$\pi_{\mathbb{C}} : \ell_2(V) \to \mathbb{C}, \qquad (\pi_{\mathbb{C}} f)(x) = \frac{1}{\#V} \sum_{x \in V} f(x),$$
$$\pi_0 : \ell_2(V) \to \ell_2^0(V), \quad (\pi_0 f)(x) = f(x) - \pi_{\mathbb{C}} f(x).$$

Thus the infimum in the Definition 5.6.1 is taken over the space $\ell_2^0(V)$. The number λ_1 defined above can be viewed as

$$\lambda_1(X) = \inf_{\ell_2^0(V)} \frac{\langle \nabla f, \nabla f \rangle}{\langle f, f \rangle} = \inf_{\ell_2^0(V)} \frac{\langle \nabla^* \nabla f, f \rangle}{\langle f, f \rangle}.$$

The operator $\nabla^* \nabla$ is the discrete Laplace operator and λ is the smallest positive eigenvalue of the discrete Laplacian on X. (See also Exercise 5.2).

Definition 5.6.2. A countable collection $\{X_n\}_{n=1}^{\infty}$, $X_n = (V_n, E_n)$, of finite graphs of bounded degree, satisfying $V_n \to \infty$, is said be a collection of *expanders* if there exists $c > 0$ such that

$$\lambda_1(X_n) \geq c \tag{5.3}$$

for every $n \in \mathbb{N}$.

The above definition of expanders will be useful for our purposes but does not convey any geometric intuition. Let us briefly describe the latter.

Definition 5.6.3. Let $X = (V; E)$ be a graph. Given a set $A \subseteq V$ define its edge boundary $\partial^e(A)$ by setting $\partial^e(A) = \{e \in E \mid e$ has exactly one vertex in $A\}$.

The edge boundary is thus the set of edges connecting a subset $A \subseteq V$ with its complement. This notion is used to define the Cheeger isoperimetric constant $h(X)$ of a finite graph $X = (V, E)$ by the formula

$$h(X) = \min \left\{ \tfrac{\#\partial^e A}{\#A} \mid 0 < \#A \leq \tfrac{\#X}{2} \right\}.$$

One then says that a sequence of graphs $\{X_i\}_{i=1}^{\infty}$ is a sequence of expanders if

$$\inf_{i \in \mathbb{N}} h(X_i) > 0.$$

In other words, expanders are finite graphs whose subsets have large boundaries. The following theorem justifies our definition of expander graphs and shows that the two definitions are equivalent.

Theorem 5.6.4. *Let X be a k-regular graph. Then* $\frac{h(X)^2}{2k} \leq \lambda_1(X) \leq 2h(X)$.

Theorem 5.6.4 was proved by Dodziuk [66], Alon [2] (left inequality) and Tanner [228], Alon and Milman [3] (right inequality). The Cheeger constant can be used to show, using counting arguments, that most large graphs are expanders; see e.g. [152], Proposition 1.2.1, or [219] for a precise statement.

It is a non-trivial fact that expanders exist. We will construct examples in the next chapter. For now we will show that geometry of expanders is highly incompatible with that of Euclidean space.

Theorem 5.6.5. *Let $\{X_n = (V_n, E_n)\}_{n\in\mathbb{N}}$ be a sequence of expanders. Then the coarse disjoint union of the X_n does not embed coarsely into a Hilbert space.*

Proof. Assume that we are given a large scale Lipschitz map f from X, the coarse disjoint union of X_n, into the Hilbert space H. We will show that this map cannot be a coarse embedding. Restricting f to V_n for every $n \in \mathbb{N}$ we obtain a map $f_n\colon V_n \to H$, which satisfies the uniform estimate

$$\|f_n(x) - f_n(y)\| \leq L d_n(x,y) + C,$$

where d_n denotes the path metric on X_n, restricted to V_n. We can also assume, by shifting the image appropriately, that each f_n satisfies $\sum_{x\in V_n} f_n(x) = 0$.

Take an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H and write each $f_n(x)$ in this basis. Let f_n^i denote the i-th coordinate of f_n, i.e., $f_n^i = \langle f_n, e_i\rangle$. Applying inequality (5.1) to f_n^i for every $i \in \mathbb{N}$ gives

$$\sum_{i=1}^{\infty}\sum_{x\in V_n} |f_n^i(x)|^2 \leq \sum_{i=1}^{\infty} \frac{1}{\lambda_1(X_n)}\Big(\sum_{x\in V_n}\sum_{y\in B(x,1)} |f_n^i(x) - f_n^i(y)|^2\Big),$$

which translates to

$$\sum_{x\in V_n} \|f_n(x)\|^2 \leq \frac{1}{\lambda_1(X_n)}\Big(\sum_{x\in V_n}\sum_{y\in B(x,1)} \|f_n(x) - f_n(y)\|^2\Big).$$

Since each f_n is large scale Lipschitz with positive constants L and C, which are independent of n, we have

$$\frac{1}{\#V_n}\sum_{x\in V_n} \|f_n(x)\|^2 \leq \frac{k(L+C)}{\lambda_1(X_n)},$$

where k is the degree of X_n. Since there are $\#V_n$ summands on the left-hand side, at least half of them cannot exceed $2k(L+C)/\lambda_1(X_n)$. In other words, f_n maps at least half of the points in each V_n into a ball of radius $2k(L+C)/\lambda_1(X_n)$. By definition of expanders, for $R = 2k(L+C)/c$, where $c > 0$ is as in (5.3), we have

$$\frac{2k(L+C)}{\lambda_1(X_n)} \leq R.$$

This shows that f is not a coarse embedding, since otherwise the cardinality of the preimage of a ball of radius at most c under each f_n would be uniformly bounded above. □

In the next chapter we will discuss explicit examples of expanders. They can be constructed as coarse disjoint unions of certain finite quotients of the groups $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 2$. However the cases $n = 2$ and $n \geq 3$ differ significantly. More precisely, the groups $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$ have Kazhdan's property (T), while $\mathrm{SL}(2, \mathbb{Z})$ has a free subgroup of finite index and a weaker property (τ) with respect to a certain family of subgroups; see [153], [152].

A similar, although slightly more complicated argument shows that expanders do not embed coarsely into any ℓ_p, $p \geq 1$ [160], [209]. Ozawa showed that expanders do not embed coarsely into any uniformly convex Banach space with an unconditional basis [196]. Moreover, V. Lafforgue constructed families of expanders which do not embed coarsely into any uniformly convex Banach space [150]. His construction uses strengthened versions of property (T). Mendel and Naor [165] gave different examples of expanders with such a non-embeddability property. Their construction uses the zig-zag product. The following question is still open.

Open question 5.6.6. *Does there exist a sequence of expanders which embeds coarsely into a uniformly convex Banach space?*

5.7 A geometric characterization of non-embeddability

Expanders are important examples of spaces which do not embed coarsely into the Hilbert space. In fact, essentially every currently known explicit example of a metric space which does not admit such a coarse embedding is constructed using expanders. One of the open questions concerning coarse embeddability is the following: *given a finitely generated group which does not embed coarsely into the Hilbert space, does it necessarily contain a sequence of expanders in some coarse way?* In this section we will give a characterization of coarse non-embeddability into a Hilbert space that is related to this problem. It was proved independently by Ostrovskii [192] and Tessera [230] and says that a space X does not coarsely embed into the Hilbert space if and only if there is a sequence of finite subsets of X which obstruct embeddability. These sets have certain features that make them similar to expanders.

Definition 5.7.1 ([230]). Let $k, r > 0$. A finite metric space (X, d) is called a *generalized* (k, r)*-expander* if there exists a probability measure μ on $X \times X$ such that

(1) $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$,

(2) $\mu(x,y) = 0$ if $d(x,y) \leq r$,
(3) for every map $f : X \to H$ to a Hilbert space satisfying

$$\|f(x) - f(y)\|_H \leq d(x,y)$$

for all $x, y \in X$, the inequality

$$\sum_{d(x,y)\geq 1} \|f(x) - f(y)\|_H^2 \mu(x,y) \leq k^2$$

holds.[2]

A sequence of finite metric spaces $\{(X_n, d_n)\}_{n=1}^{\infty}$ is called a sequence of generalized k-expanders if there exists a sequence $r_n \to \infty$ such that for every $n \in \mathbb{N}$ the space X_n is a generalized (k, r_n)-expander.

Another way to state the third condition is to say that X satisfies a sequence of Poincaré-type inequalities. Poincaré inequalities arise in many contexts in geometry and geometric function theory. In the Euclidean setting they are of the form

$$\int_B |u - u_B|^p \,\mathrm{dx} \leq C(\operatorname{diam} B)^p \int_B |\nabla u|^p,$$

where $p \geq 1$, u belongs to a certain Sobolev space on $\mathbb{R}^n$, C is a constant dependent on n and p, ∇ is the gradient, B is some fixed ball and

$$u_B = \int_B u(x)\,\mathrm{dx}$$

denotes the mean value of u over B. The condition we used to introduce expanders in Definition 5.6.2 is also a Poincaré-type inequality since in the definition of λ_1 we already assumed that the mean of f vanishes. See [125] for a discussion of Poincaré inequalities in the context of Sobolev spaces and metric geometry. We will first investigate in what sense generalized expanders are similar to expanders.

Proposition 5.7.2. *A sequence of expanders of bounded degree is a sequence of generalized expanders.*

Proof. Let $X = (V, E)$ be a finite graph, H a Hilbert space and let $\lambda_1 = \lambda_1(X)$. Inequality (5.3) in Definition 5.6.2 implies that

$$\sum_{x\in V} \|f(x)\|^2 \leq \frac{1}{\lambda_1} \sum_{d(x,y)\leq 1} \|f(x) - f(y)\|^2 \tag{5.4}$$

[2]The condition $d(x,y) \geq 1$ in the above sum can be replaced by $d(x,y) \geq C$ for other constants $C > 0$. In fact, we will be mainly concerned with the case $d(x,y) \to \infty$.

for $f: V \to H$, assuming $\sum_{x \in X} f(x) = 0$. For such f we have

$$\begin{aligned}
&\sum_{x,y \in V} \|f(x) - f(y)\|^2 \\
&\quad = \sum_{x \in V} \sum_{y \in V} (\|f(x)\|^2 + \|f(y)\|^2 - 2\langle f(x), f(y)\rangle) \\
&\quad = \sum_{x \in V} \Big(\#V \|f(x)\|^2 + \sum_{y \in V} \|f(y)\|^2 \Big) - \Big\langle \sum_{x \in V} f(x), \sum_{y \in V} f(y) \Big\rangle \\
&\quad = 2\#V \sum_{x \in V} \|f(x)\|^2.
\end{aligned}$$

This, combined with inequality (5.4), shows that (5.4) is equivalent to the inequality

$$\frac{1}{\#V^2} \sum_{x,y \in V} \|f(x) - f(y)\|^2 \leq \frac{1}{\lambda_1 \#V} \sum_{x \in V} \sum_{y \in B(x,1)} \|f(x) - f(y)\|^2.$$

Let μ denote the uniform measure on $V \times V$,

$$\mu(x, y) = \frac{1}{\#V^2}$$

for all $(x, y) \in V \times V$. Then

$$\sum_{x,y \in V} \|f(x) - f(y)\|^2 \mu(x, y) \leq \frac{1}{\lambda_1 \#V} \sum_{x \in V} \sum_{y \in B(x,1)} \|f(x) - f(y)\|^2.$$

We can assume that f is 1-Lipschitz, which implies that

$$\sum_{x,y \in V} \|f(x) - f(y)\|^2 \mu(x, y) \leq \frac{1}{\lambda_1 \#V} (\#V k) = \frac{k}{\lambda_1},$$

where k is the upper bound on the degree of each vertex.

Let $\{X_n\}$ be a sequence of expanders. We need to show that for each X_n, μ_n as above can be replaced by a measure supported on a complement of a bounded neighborhood of the diagonal in $V_n \times V_n$. Let

$$R_n = \log_k \left(\frac{\#V_n}{2} \right).$$

Then the uniform measure satisfies

$$\sum_{d(x,y) \geq R_n} \mu_n(x, y) \geq \frac{1}{2}.$$

Indeed, if $R_n > 0$, then the number of elements $(x, y) \in V_n \times V_n$ satisfying $d(x, y) \leq R_n$ is at most $k^{R_n} \# V_n$. It is also clear that R_n tends to infinity. Denote by ν_n the normalized restriction of μ_n to the complement of the R_n-neighborhood of the diagonal. Then

$$\sum_{x,y \in V_n} \|f(x) - f(y)\|^2 \nu_n(x, y) \leq \frac{k}{\lambda_1(X_n)},$$

as required. □

The main theorem we will prove in this section is the following.

Theorem 5.7.3. *A quasi-geodesic metric space X does not coarsely embed into a Hilbert space if and only if there exists a sequence of generalized expanders $\{(X_n, d_n)\}$ whose coarse disjoint union embeds coarsely into X.*

Note that there is no assumption of local finiteness or bounded geometry. We will need the following lemma.

Lemma 5.7.4. *Let X be a metric space, $x_0 \in X$ be a distinguished point and let $\rho_+ \colon [0, \infty) \to [0, \infty)$ be a non-decreasing function satisfying* $\lim_{t \to \infty} \rho_+(t) = \infty$. *Assume that for every $n \in \mathbb{N}$ there exists $\kappa > 0$ such that for all $i, j \geq \kappa$ there is a map $f_i \colon B(x_0, i) \to H$ satisfying*

$$\|f_i(x) - f_i(y)\| \leq \rho_+(d(x, y))$$

for $x, y \in B(x_0, j)$ and

$$\|f_i(x) - f_i(y)\| \geq n$$

for $x, y \in B(x_0, i)$ whenever $\kappa \leq d(x, y) \leq j$. Then X admits a coarse embedding into a Hilbert space.

Proof. The construction of the embedding is similar to the one given in Theorem 5.2.8. Let $R > 0$ and let m_n be an increasing sequence such that

$$\sum_{n=1}^{\infty} \left(\frac{1}{m_n}\right)^2 \leq 1,$$

and for which there is a constant $C > 0$ such that

$$\frac{n}{m_n} \geq C$$

for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exists κ_n as above and we can assume that the sequence $\{\kappa_n\}$ is increasing. For each $n \in \mathbb{N}$ let $j_n \geq \max\{\kappa_{3n}, R\}$. Let

$f_n : B(x_0, j_n) \to H$ be as above and define $f : B(x_0, R) \to (\bigoplus_{k=1}^{\infty} H)_{(2)}$ by setting

$$f(x) = \bigoplus_{n=1}^{\infty} \tfrac{1}{m_n}(f_n(x) - f_n(x_0)).$$

Note that if we define

$$\rho_-(t) = C\,\#\{n \in \mathbb{N} \mid \kappa_n \leq t \leq j_n\},$$

then f satisfies

$$\rho_-(d(x,y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(x,y)).$$

Indeed,

$$\|f(x) - f(y)\| = \sqrt{\sum_{n=1}^{\infty} \left(\frac{1}{m_n}\right)^2 \|f_n(x) - f_n(y)\|_H^2} \leq \rho_+(d(x,y))$$

by the assumption on m_n, while the lower estimate follows from the properties of the functions f_n.

We need to verify that $\rho_-(t) \to \infty$ as $t \to \infty$. For this note that the quantity

$$\#\{n \in \mathbb{N} \mid \kappa_n \leq t\}$$

increases to infinity with t. If we let $N = \rho_-(t)$, then $\kappa_N \leq t \leq \kappa_{N+1}$. As t is increasing, we have

$$\kappa_{[N/2]} \leq t \leq \kappa_{N+1} \leq \kappa_{[3N/2]}$$

for large N. Hence,

$$\kappa_n \leq t \leq \kappa_{3n}$$

whenever $\frac{N}{2} \leq n \leq N$, which gives

$$\rho_-(t) \geq \frac{1}{2}\#\{n \mid \kappa_n \leq t\}.$$

In this way we obtain an embedding of the ball $B(x_0, R)$ for every radius $R > 0$. Since ρ_- and ρ_+ do not depend on R, the assertion now follows from Theorem 5.5.1. □

We are now in a position to prove the main theorem.

Proof of Theorem 5.7.3. Assume that $f : X \to H$ is a coarse embedding of X into the Hilbert space, satisfying

$$\rho_-(d(x,y)) \leq \|f(x) - f(y)\|_H \leq d(x,y).$$

(Note that we can, without loss of generality, restrict the embedding to a discrete net in X and rescale the image, to obtain the upper bound). Then

$$\begin{aligned}\sum_{d(x,y)\geq 1} \|f(x)-f(y)\|_H^2 \mu(x,y) &\geq \sum_{d(x,y)\geq 1} \rho_-(d(x,y))^2 \mu(x,y)\\ &\geq \sum_{d(x,y)\geq R} \rho_-(R)^2 \mu(x,y)\\ &\geq \rho_-(R)^2\end{aligned}$$

if μ is such that $\mu(x,y)=0$ whenever $d(x,y)\leq R$. Since R tends to infinity and $\lim_{t\to\infty}\rho_-(t)=\infty$, we obtain a contradiction with the definition of generalized expanders.

To prove the other implication assume that the metric space X does not coarsely embed into a Hilbert space. By Lemma 5.7.4, there exists $N>0$ such that for all $\kappa>0$ there are $i,j\geq\kappa$ such that for every map $f\colon B(x_0,i)\to H$ satisfying

$$\|f(x)-f(y)\|\leq \rho_+(d(x,y))$$

there exist $x,y\in B(x_0,i)$, $\kappa\leq d(x,y)\leq j$, for which

$$\|f(x)-f(y)\|\leq N.$$

The generalized expanders will be constructed out of balls $B(x_0,i)$.

Let A_1 be the set of functions $\eta\colon B(x_0,i)\times B(x_0,i)\to[0,\infty)$, restricted to the set $\{(x,y)\mid \kappa\leq d(x,y)\leq j\}$, satisfying

$$\eta(x,y)\leq\rho_+^2(d(x,y))$$

for $x,y\in B(x_0,i)$ and

$$\eta(x,y)\geq N$$

for $x,y\in B(x_0,i)$ such that $\kappa\leq d(x,y)\leq j$.

Let A_2 be the restriction to the set $\{(x,y)\mid\kappa\leq d(x,y)\leq j\}$ of functions $\eta\colon B(x_0,i)\times B(x_0,i)\to[0,\infty)$ which are of the form

$$\eta(x,y)=\|f(x)-f(y)\|^2$$

for some $f\colon B(x_0,i)\to H$ satisfying

$$\|f(x)-f(y)\|\leq\rho_+(d(x,y)).$$

The sets A_1 and A_2 are convex subsets of $\ell_2(B(x_0,i)\times B(x_0,i))$. By Lemma 5.7.4, they are also disjoint. By the Hahn–Banach theorem, there exists a functional φ separating A_1 and A_2; that is, φ is non-negative on A_1 and non-positive on A_2.

Since every functional on a Hilbert space is represented by the inner product with a fixed vector,[3] there exists $\nu \in \ell_2(B(x_0, i)^2)$ such that

$$\langle \eta, \nu \rangle \geq 0 \tag{5.5}$$

whenever $\eta \in A_1$ and

$$\langle \eta, \nu \rangle \leq 0 \tag{5.6}$$

whenever $\eta \in A_2$.

Now let $\nu_+(x, y) = \max\{\nu(x, y), 0\}$ and $\nu_- = \max\{-\nu(x, y), 0\}$ so that $\nu(x, y) = \nu_+(x, y) - \nu_-(x, y)$. Let

$$\eta(x, y) = \begin{cases} N^2 & \text{if } \nu(x, y) > 0 \text{ and } d(x, y) \geq \kappa, \\ 0 & \text{if } \nu(x, y) > 0 \text{ and } d(x, y) < \kappa, \\ \rho_+^2(d(x, y)) & \text{otherwise.} \end{cases}$$

Inequality (5.5), applied to η, yields

$$\sum_{x,y} \rho_+^2(d(x, y))\nu_-(x, y) \leq N^2 \sum_{d(x,y)\geq\kappa} \nu_+(x, y). \tag{5.7}$$

Now consider $\zeta(x, y) = \|f(x) - f(y)\|^2$ for some map $f : X \to H$ into a Hilbert space H satisfying

$$\|f(x) - f(y)\| \leq \rho_+(d(x, y)).$$

Then, by (5.6),

$$\begin{aligned} \sum_{x,y} \|f(x) - f(y)\|^2 \nu_+(x, y) &\leq \sum_{x,y} \|f(x) - f(y)\|^2 \nu_-(x, y) \\ &\leq \sum_{x,y} \rho_+^2(d(x, y))\nu_-(x, y). \end{aligned}$$

The last terms can be now estimated using inequality (5.7),

$$\sum_{x,y} \|f(x) - f(y)\|^2 \nu_+(x, y) \leq N^2 \sum_{d(x,y)\geq\kappa} \nu_+(x, y).$$

Thus if we define

$$\mu(x, y) = \begin{cases} \frac{\nu_+(x,y)}{\sum_{d(x,y)\geq\kappa} \nu_+(x,y)} & \text{if } d(x, y) \geq \kappa, \\ 0 & \text{otherwise,} \end{cases}$$

then the theorem follows. (To check that $\sum_{d(x,y)\geq\kappa} \nu_+(x, y)$ is not zero is left as an exercise.) □

[3] By the Riesz representation theorem, see e.g. [215].

The following question is open.

Open question 5.7.5. *Let X be a bounded geometry metric space which does not embed coarsely into the Hilbert space. Does X coarsely contain expanders?*

For spaces which are not locally finite the answer is known to be negative. It was shown by W. B. Johnson and N. L. Randrianarivony [138] that the infinite-dimensional space ℓ_p does not embed coarsely into the Hilbert space when $p > 2$. On the other hand it is known that expanders do not embed coarsely into L_p spaces for $p \geq 1$, see [160], [209]. Observe that the above discussion also leads to the conclusion that generalized expanders *do* embed into ℓ_p, $p > 2$.

5.8 Compression of coarse embeddings

So far in this chapter we were only interested in the fact whether a given metric space embeds coarsely into a Hilbert space or not, but if the answer is affirmative, it might be worth asking *how well does it embed?* To give a quantitative measure of coarse embeddability we will study the lower bound ρ_-. The following notion of a compression number was introduced by Guentner and Kaminker [115], extending a definition of Gromov [108].

Definition 5.8.1. Let X be a quasi-geodesic metric space. We define the *Hilbert space compression* of X, denoted by $\mathcal{R}_X$, as the supremum of all $\alpha \in [0, 1]$ for which there exists a coarse embedding $f : X \to H$ into a Hilbert space H such that $Ct^{\alpha} \leq \rho_-(t)$ for some constant $C > 0$ and all $t > K$, for some $K \geq 0$.

Recall that the upper bound ρ_+ can always be chosen to be linear, since X is quasi-geodesic.

Example 5.8.2. If X embeds isometrically into a Hilbert space then the Hilbert space compression is $\mathcal{R}_X = 1$, since $\rho_-(t) = t$. The same conclusion holds for spaces which embed quasi-isometrically into a Hilbert space.

The reader might observe that in the case of a standard embedding of a tree, constructed in Example 5.2.2, the lower bound satisfies $\rho_-(t) \geq C\sqrt{t}$, which gives a lower bound of $1/2$ on the compression of T. But it turns out that it is possible to construct embeddings of a tree with better ρ_-.

Theorem 5.8.3. *Let T be a tree. Then the Hilbert space compression satisfies $\mathcal{R}_T = 1$.*

Proof. We will modify the construction from Example 5.2.2 and show that for every $\alpha < 1$ there exists a coarse embedding for which

$$\rho_-(t) \geq Ct^{\alpha}$$

for some constant $C > 0$.

Fix a point r_0 in T and consider the Hilbert space $\ell_2(V)$, where V is the set of vertices of T. For a parameter $\beta \in (0, \frac{1}{2})$ construct an embedding $F_\beta \colon V(T) \to \ell_2(V(T))$ as follows. For a vertex x satisfying $d(x, r_0) = n$, let $x = x_0, x_1, \ldots, x_{n-1}, x_n = r_0$ be the vertices on the unique geodesic path linking x and r_0 such that $d(x_k, x_{k+1}) = 1$. Let

$$F_\beta(x) = \sum_{k=0}^{d(r_0,x)} k^\beta 1_{x_k}.$$

To show that F_β is large scale Lipschitz it is enough to restrict to pairs of points x and y satisfying $d(x, y) \leq 1$, since T is quasi-geodesic. Note that if $d(x, y) \leq 1$, then we can assume without loss of generality that $d(r_0, x) = d(r_0, y) + 1$, and that if $x_0, \ldots, x_n$ are the vertices on the geodesic path from x to r_0 as before, then $y = x_1$. In that case,

$$\|F_\beta(x) - F_\beta(y)\|^2 = 1^\beta + \sum_{k=1}^{d(r_0,y)} (k^\beta - (k-1)^\beta)^2.$$

In order to show that F_β is indeed a coarse map we need to show that the right-hand side of the above equation is finite and independent of x and y. If we set $g_\beta(t) = t^\beta$, then

$$\begin{aligned}\sum_{k=1}^{\infty}(k^\beta - (k-1)^\beta)^2 &= \sum_{k=1}^{\infty}\Big(\int_{k-1}^{k} g'_\beta(t)\,dt\Big)^2 \\ &\leq \sum_{k=1}^{\infty}\int_{k-1}^{k}(g'_\beta(t))^2\,dt = \beta^2\int_0^\infty t^{2\beta-2}\,dt = \frac{\beta^2}{1-2\beta},\end{aligned}$$

since $\beta < 1/2$ is fixed. Thus F_β is large scale Lipschitz.

It remains to compute ρ_-. Let $x, y \in T$ satisfy $d(x, y) \geq R$. Without loss of generality we can assume that $d(x, r_0) \geq d(y, r_0)$. Note that if $\{x_k\}_{k=0}^{d(x,r_0)}$ and $\{y_k\}_{k=0}^{d(y,r_0)}$ are the vertices on the unique geodesic paths from x and y, respectively, to r_0, as before, then the sets $\{x_k \mid k = 1, \ldots, \lceil R/2 \rceil\}$ and $\{y_k\}_{k=0}^{d(y,r_0)}$ are disjoint, without loss of generality. Thus

$$\begin{aligned}\|F_\beta(x) - F_\beta(y)\|^2 &\geq \sum_{k=0}^{\lceil R/2 \rceil} k^{2\beta} \\ &\geq \int_0^{R/2}(g'_\beta(t))^2\,dt = \frac{R^{2\beta+1}}{(2^{2\beta+1})(2\beta+1)} = C(R^{1/2+\beta})^2.\end{aligned}$$

Hence F_β is a coarse embedding satisfying

$$\rho_-(t) \geq C' t^{1/2+\beta}.$$

Since the estimate holds for any $\beta \in (0, 1/2)$, the compression, as the supremum of such β, is equal to 1. □

It is important to point out that $\mathcal{R}_G = 1$ is not equivalent to the existence of a quasi-isometric embedding. The case of a tree illustrates this phenomenon. One can deduce from a result of Bourgain [36] that trees do not embed quasi-isometrically into a Hilbert space, or more generally, a uniformly convex Banach space, even though their Hilbert space compression is 1.

5.9 Compression $> \frac{1}{2}$ implies property A

Theorems 5.3.1 and 4.5.3 imply that, in general, property A is not equivalent to coarse embeddability into the Hilbert space. However, it turns out that for bounded geometry metric spaces, a sufficiently good embedding implies property A.

Theorem 5.9.1 ([115]). *Let G be a finitely generated group with a word length metric. If the Hilbert space compression satisfies $\mathcal{R}_G > \frac{1}{2}$, then G has property A.*

We need to assume the basic theory of bounded linear operators on Hilbert spaces and C*-algebras, in particular we will make use of the notion of a positive operator and its square root. The Hilbert spaces we consider in this section are over the complex numbers. We refer to [67], [201], [214], [245] for background on functional analysis and the theory of operators.

Recall that given a Hilbert space H, the algebra of bounded operators on H is denoted by $B(H)$. Recall that the operator norm is defined by

$$\|T\| = \sup\{\|Tv\|_H \mid v \in H,\ \|v\| = 1\}$$

for $T \in B(H)$.

To prove the theorem we will need to introduce the uniform Roe algebra. The uniform Roe algebra is a metric, "non-equivariant" version of the reduced C*-algebra $C_r^*(G)$ of a group. In our case it will allow the use of approximation arguments. We will construct the algebra for a uniformly discrete metric space, however it is possible to give a more general definition. Let X be a uniformly discrete metric space. A linear operator T on the Hilbert space $\ell_2(X)$ can be represented as an infinite matrix, indexed by $X \times X$, with entries $T_{(x,y)} = \langle T1_x, 1_y\rangle$.

Definition 5.9.2. Let T be a linear operator on $\ell_2(X)$. The *propagation* of T, denoted by $p(T)$, is the smallest number $R \geq 0$ such that $T_{(x,y)} = 0$ for all pairs (x, y) satisfying $d(x, y) \geq R$.

Note that in this setting pairs of points $d(x,y) \le R$ are exactly the ones on the strip of width R along the main diagonal $(x,x) \in X \times X$. Thus, if T has finite propagation, then the matrix coefficients vanish everywhere, except some R-neighborhood of the main diagonal. Consider the space of finite propagation operators,

$$\mathbb{C}_u X = \{T \in B(\ell_2(X)) \mid p(T) < \infty\}.$$

We leave it as an exercise to verify that $\mathbb{C}_u X$ is in fact an algebra under the usual matrix multiplication. Clearly, the identity operator also belongs to $\mathbb{C}_u X$.

Definition 5.9.3. The *uniform Roe algebra* $C_u^*(X)$ is the closure of $\mathbb{C}_u X$ in the operator norm in $B(\ell_2(X))$.

The outline of the proof of Theorem 5.9.1 is as follows. Given a coarse embedding we obtain functions ξ into the unit sphere of a Hilbert space, as in Theorem 5.2.8. Then we construct a matrix by defining $T_{(x,y)} = \langle \xi_x, \xi_y \rangle$. If this matrix is a positive continuous operator on $\ell_2(X)$, then, by spectral calculus, we can take its positive square root $Q = Q^*$ such that $Q^2 = QQ^* = T$. Additionally, if the compression is at least $1/2+\varepsilon$, then T, and consequently Q, belong to $C_u^*(X)$. This means that if we denote by Q_x the vector $Q1_x$, then we have $\langle Q_x, Q_y \rangle = \langle \xi_x, \xi_y \rangle$. Then, using the fact that $Q \in C_u^*(X)$, we will approximate Q_x by functions which have uniformly bounded supports, and derive property A.

Assume that $f\colon X \to H$ is a coarse embedding into a Hilbert space H, with functions ρ_- and ρ_+ as in Definition 5.1.1. Then, by Theorem 5.2.8, for every $t > 0$ there is a map $\xi\colon X \to \exp(H)$ into the Hilbert space $\exp(H)$ such that

$$\exp(-t\rho_+(d(x,y))^2) \le \langle \xi_x, \xi_y \rangle \le \exp(-t\rho_-(d(x,y))^2).$$

In particular, $\langle \xi_x, \xi_x \rangle = 1$ for every $x \in X$ and $\langle \xi_x, \xi_y \rangle$ tends to 0 uniformly as $d(x,y) \to \infty$. We define a matrix $[T_{(x,y)}]$, where $(x,y) \in X \times X$, by setting

$$T_{(x,y)} = \langle \xi_x, \xi_y \rangle. \tag{5.8}$$

The Schur test gives a sufficient condition for an operator induced by a matrix to be bounded.

Proposition 5.9.4 (Schur test). *Let T be an infinite matrix with entries $T_{(x,y)}$ such that $T_{(x,y)} \ge 0$ for every $x, y \in X$ and satisfying*

$$\sup_{x\in X} \sum_{y\in X} T_{(x,y)} = C_1 < \infty \quad \textit{and} \quad \sup_{y\in X} \sum_{x\in X} T_{(x,y)} = C_2 < \infty.$$

Then the linear operator given by T is bounded on $\ell_2(X)$ and $\|T\| \le \max\{C_1, C_2\}$.

The proof rests on the Cauchy–Schwartz inequality, we refer to [121] for the argument. In our setting, a coarse embedding with ρ_- growing sufficiently fast gives rise to a matrix T which satisfies the above conditions.

Proposition 5.9.5. *Let G be a finitely generated group and $f : X \to H$ be a coarse embedding into a Hilbert space. If there exist $\varepsilon > 0$, $C > 0$ and $D > 0$ such that*

$$\rho_-(t) \geq Ct^{1/2+\varepsilon}$$

for every $t \geq D$, then the operator induced by the matrix T, defined as in (5.8), is bounded and belongs to the uniform Roe algebra $C_u^*(X)$.

Proof. Choose a finite generating set Σ of G. Let $t > 0$ and consider the operator T as above. By the assumption on compression,

$$\rho_-^2(d(x,y)) \geq Cd(x,y)^{1+\delta}$$

for some $C > 0$ and $\delta > 0$. We now apply the Schur test. First note that it is enough to check only one of the conditions since $T_{(x,y)} = T_{(y,x)}$[4] by the definition of T. Let $y \in G$ be fixed. We have

$$\begin{aligned}\sum_{x\in G} T_{(x,y)} &= \sum_{m\geq 0} \sum_{d(x,y)=m} T_{(x,y)} \\ &= \sum_{m\leq k} \sum_{d(x,y)=m} T_{(x,y)} + \sum_{m>k} \sum_{d(x,y)=m} T_{(x,y)},\end{aligned} \tag{5.9}$$

where k will be chosen later. The first sum is a finite sum in which each term is at most 1 and their number depends only on k. Hence it will be sufficient to show that the second sum is finite. For the purposes of the next calculation we denote by $S(x,m)$ the sphere of radius m centered at x; that is, the set of points, whose distance from x is exactly m. We have

$$\#S(x,m) \leq \#(\Sigma^m) \leq (\#\Sigma)^m.$$

It follows from the estimate on ρ_- and the previous remarks that

$$\begin{aligned}\sum_{m>k} \sum_{d(x,y)=m} T_{(x,y)} &= \sum_{m>k} \sum_{d(x,y)=m} \exp(-t\rho_-^2(d(x,y))) \\ &= \sum_{m>k} \sum_{d(x,y)=m} \exp(-tm^{1+\delta}) \\ &= \sum_{m>k} \#S(x,m)\exp(-tm^{1+\delta})\end{aligned}$$

[4]This condition holds since the Hilbert space we use for the embedding is a real Hilbert space. However we could equivalently consider an embedding into a complex Hilbert space, since embeddability properties with the given ρ_- and ρ_+ into a real and complex Hilbert spaces are equivalent.

$$
\begin{aligned}
&\leq \sum_{m>k} \frac{\#\Sigma^m}{\exp(tm^{1+\delta})} \\
&= \sum_{m>k} \left(\frac{\#\Sigma}{\exp(tm^{\delta})} \right)^m \\
&\leq \sum_{m>k} \left(\frac{\#\Sigma}{\exp(tk^{\delta})} \right)^m .
\end{aligned}
$$

Now we choose k large enough to guarantee

$$\left(\frac{\#\Sigma}{e^{tk^{\delta}}} \right) < 1,$$

in which case we have

$$\sum_{m>k} \left(\frac{\#\Sigma}{\exp(tk^{\delta})} \right)^m = \left(\frac{\#\Sigma}{\exp(tk^{\delta})} \right)^{k+1} \left(1 - \left(\frac{\#\Sigma}{\exp(tk^{\delta})} \right) \right)^{-1} . \tag{5.10}$$

We conclude from (5.10) that the sum (5.9) converges. Note also that the above argument is independent of the choice of $y \in G$, which proves that the operator is bounded.

Let us denote by T_k the operator T restricted to the k-neighborhood of the diagonal, given by the matrix

$$(T_k)_{(x,y)} = \begin{cases} T_{(x,y)} & \text{if } d(x,y) \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the Schur test applied to $T - T_k$ and the estimate (5.10) show that

$$\|T - T_k\| \leq \sum_{m>k} \left(\frac{\#\Sigma}{\exp(tk^{\delta})} \right)^m \to 0$$

as $k \to \infty$. Thus the operator T is an element of the uniform Roe algebra $C_u^*(G)$ since it is a limit (in the operator norm) of finite propagation operators T_k. □

An operator T on a Hilbert space H is *positive* if $\langle Tv, v \rangle \geq 0$ for every $v \in H$. Positive operators share many properties of positive numbers. The feature that we will need is that they have square roots. More precisely, for any positive operator $T \in A$, where $A \subseteq B(H)$ is a C*-subalgebra of $B(H)$, there exists a positive operator $Q \in A$ such that

$$QQ^* = T.$$

The operator Q is called the *positive square root* of T and its existence can be proven using the Spectral Theorem (see e.g. [67], [214]). We would like to apply this discussion to the operator T. As we have seen earlier, matrices of the form $\langle \xi_x, \xi_y \rangle$ are positive definite kernels. This yields positivity of the operator induced in this way.

Proposition 5.9.6. *Let X be a countable set and let $\xi\colon X \to H$ be a map which satisfies $\|\xi_x\|_H = 1$ such that the matrix $T_{(x,y)} = \langle \xi_x, \xi_y \rangle$ induces a bounded linear operator on $\ell_2(X)$. Then T is a positive operator.*

We can now prove the main result of this section.

Proof of Theorem 5.9.1. Let $F\colon G \to H_0$ be a coarse embedding. Given $R > 0$ and $\varepsilon > 0$ construct, as before, a function $\xi\colon G \to H$ into the unit sphere of a Hilbert space H satisfying the conditions of Proposition 5.2.8. Let $T_{(x,y)} = \langle \xi_x, \xi_y \rangle$. The operator T is positive and belongs to the uniform Roe algebra $C_u^*(G)$ by Propositions 5.9.5 and 5.9.6. Hence there exists a positive square root $Q \in C_u^*(G)$ such that $Q^*Q = T$. The operator Q is again represented by a matrix, indexed by $G \times G$, and it is a limit, in the operator norm, of operators of bounded propagation. We choose $U \in C_u^*(G)$ so that U has finite propagation and

$$\|Q - U\| \leq \varepsilon.$$

We now define for every $x \in G$ the vector $\eta_x \in \ell_2(G)$,

$$\eta_x = U 1_x,$$

the x-column of U. In this way we obtain an approximation in the Hilbert space norm,

$$\|\eta_x - Q1_x\|_{\ell_2(G)} \leq \|U - Q\| \, \|1_x\|_{\ell_2(G)} \leq \varepsilon \tag{5.11}$$

for every $x \in G$. Observe that

$$\|Q1_x\|_{\ell_2(G)} = \langle Q1_x, Q1_x \rangle = T_{(x,x)} = 1.$$

It follows that

$$|1 - \|\eta_x\|_H| \leq \varepsilon. \tag{5.12}$$

By inequality (5.11), whenever $d(x,y) \leq R$ we have

$$\begin{aligned} \|\eta_x - \eta_y\|_H &\leq \|\eta_x - Q1_x\|_H + \|Q1_x - Q1_y\|_H + \|Q1_y - \eta_y\|_H \\ &\leq 2\varepsilon + \|Q1_x - Q1_y\|_H, \end{aligned}$$

where the last term satisfies

$$\begin{aligned} \|Q1_x - Q1_y\|_H^2 &= 2(1 - \langle Q1_x, Q1_y \rangle) \\ &= 2(1 - \langle Q^*Q1_x, 1_y \rangle) \\ &= 2(1 - \langle T1_x, 1_y \rangle) \\ &= 2(1 - T_{(x,y)}) \\ &\leq 2(1 - \exp(-t\rho_-^2(d(x,y))) \leq 2\varepsilon, \end{aligned}$$

by construction and the choice of t depending on R and ε (as in the proof of Theorem 5.2.8). Consequently,

$$\|\eta_x - \eta_y\|_H \leq 4\varepsilon$$

when $d(x, y) \leq R$. We are almost done, except for the fact that the norm of η_x is not necessarily 1 for every $x \in G$. We will now normalize the η_x and show that they retain the required properties. Define

$$\xi_x = \frac{\eta_x}{\|\eta_x\|_H}.$$

Then $\|\xi_x\| = 1$ and

$$\|\xi_x - \eta_x\|_H = \left\| \frac{\eta_x}{\|\eta_x\|_H} - \eta_x \right\|_H = \left| 1 - \frac{1}{\|\eta_x\|} \right| \|\eta_x\|_H \leq \left| 1 - \frac{1}{\|\eta_x\|} \right| (1 + \varepsilon)\,.$$

Inequality (5.12) gives

$$1 - \frac{1}{1-\varepsilon} \leq 1 - \frac{1}{\|\eta_x\|_H} \leq 1 - \frac{1}{1+\varepsilon},$$

which, after applying the absolute value and simplifying, leads to the estimate

$$\left| 1 - \frac{1}{\|\eta_x\|_H} \right| \leq \max\left\{ \frac{\varepsilon}{1-\varepsilon}, \frac{\varepsilon}{1+\varepsilon} \right\} = \frac{\varepsilon}{1-\varepsilon}.$$

If $d(x, y) \leq R$, then

$$\|\xi_x - \xi_y\|_H \leq \|\xi_x - \eta_x\|_H + \|\eta_x - \eta_y\|_H + \|\eta_y - \xi_y\|_H \leq 4\varepsilon + 2\varepsilon\frac{1+\varepsilon}{1-\varepsilon}$$

by the previous calculations. Clearly, $4\varepsilon + 2\varepsilon\frac{1+\varepsilon}{1-\varepsilon}$ tends to 0 with ε.

Additionally, since Q has finite propagation, we conclude that there is a constant $D > 0$ such that $\operatorname{supp} \xi_x \subseteq B(x, D)$.

It remains to prove that we can find ξ as above, where the Hilbert space norm estimates are replaced by ℓ_1-norm estimates. This can be easily done by applying the Mazur map. We leave it to the reader to verify that defining $\xi_x' = M_{2,1}\xi_x$ satisfies the Higson–Roe condition. □

It is worth noting that one can require slightly less than compression strictly larger than $\frac{1}{2}$ in Theorem 5.9.1. It is enough to assume that the function ρ_- grows appropriately faster than the square root function, so that the Schur test still applies. For some groups the best possible compression has been computed; see e.g. [9], [38], [50]. It is not true in general that the compression of a group which embeds coarsely into the Hilbert space is greater than zero. Examples illustrating this phenomenon were constructed in [8]. In the same work for every $0 < \alpha < 1$ the authors construct a group whose Hilbert space compression is exactly α.

We will end this chapter with an open question. In 1965, R. Thompson constructed a finitely generated group which is now known as *Thompson's group F*. There are two common presentations which describe F: one finite,

$$F = \langle a, b \mid [ab^{-1}, a^{-1}ba],\ [ab^{-1}, a^{-2}ba^2] \rangle,$$

where $[a, b] = a^{-1}b^{-1}ab$ denotes the commutator of a and b; and one infinite,

$$F = \langle x_0, x_1, x_2, \dots \mid x_k^{-1} x_n x_k = x_{n+1} \ \text{ for all } \ k < n \rangle.$$

The group F can also be realized as a group of piecewise linear homeomorphisms of the unit interval $[0, 1]$ that are differentiable except at finitely many dyadic numbers and such that on intervals of differentiability the derivatives are constants and are powers of 2. See [51] for details.

The group F has many remarkable properties, yet many questions about it remain unsettled. Among the latter, the most famous one is whether F is amenable. The following problem is also interesting.

Open question 5.9.7. *Does Thompson's group F have property A?*

One potential strategy of proving that F does have property A is via Theorem 5.9.1. However, it was proved by Arzhantseva, Guba and Sapir [9] that $\mathcal{R}_F = \frac{1}{2}$. It is not known whether one could find an embedding of F for which ρ_- grows slower than t^α for any $\alpha \geq \frac{1}{2}$ but faster than $t^{1/2}$.

Exercises

Exercise 5.1. Let E be a Banach space. Assume that a sequence $\{X_n\}_{n\in\mathbb{N}}$ of finite metric spaces embeds coarsely with uniform bounds; that is, there exist embeddings $f_n \colon X_n \to E$ and non-decreasing functions $\rho_-, \rho_+ \colon [0, \infty) \to [0, \infty)$ such that

$$\rho_-(d_n(x, y)) \leq \|f_n(x) - f_n(y)\| \leq \rho_+(d_n(x, y))$$

for all $x, y \in X_n$. Prove that the coarse disjoint union of X_n embeds coarsely into $(\bigoplus_{n=1}^{\infty} E)_{(2)}$.

Exercise 5.2. In (5.2) we defined the discrete gradient ∇ on a graph. The discrete Laplacian is defined as $\Delta = \nabla^*\nabla$. Prove the formula for the discrete Laplace operator:

$$(\Delta f)(x) = \sum_{y \in B(x,1)} (f(y) - f(x)).$$

Exercise 5.3. Use the construction presented in Example 5.2.2 to show that a tree embeds isometrically into ℓ_1.

Exercise 5.4. Let G be a group and H be a subgroup of finite index. Show that Hilbert space compression satisfies $\mathcal{R}_G = \mathcal{R}_H$.

Exercise 5.5. More generally, show that Hilbert space compression is invariant under quasi-isometries.

Exercise 5.6. Let X and Y be uniformly discrete metric spaces which embed coarsely into the Hilbert space. Estimate the compression $\mathcal{R}_{X\times Y}$ in terms of $\mathcal{R}_X$ and $\mathcal{R}_Y$ if the product is equipped with the metric

$$d((x,y),(x'y')) = (d(x,x')^p + d(y,y')^p)^{1/p}$$

for any $1 \leq p < \infty$.

Exercise 5.7. Let A be a subset of a set X. Show that, given a positive-definite kernel K_A on A, the kernel defined by

$$K(x,y) = \begin{cases} K_A(x,y) & \text{if } x, y \in A, \\ 0 & \text{otherwise}, \end{cases}$$

is positive definite on X.

Exercise 5.8. Let K_β be a pointwise convergent net of positive negative definite kernels on X. Show that the limit is also a positive definite kernel.

Notes and remarks

Coarse embeddings were introduced by Gromov as *uniform embeddings* in [108]. In Banach space geometry a uniform embedding is an embedding which is a uniformly continuous homeomorphism with a uniformly continuous inverse. For a description of these uniform embeddings see [31]. For some interesting interaction between the two notions see [138], [181], [205]. Dranishnikov, Gong, Lafforgue and Yu gave the first example of a separable metric space which is not coarsely embeddable into a Hilbert space, answering negatively a question of Gromov, if every separable metric space admits such an embedding [75].

Significant interest in coarse embeddings followed [250], where the coarse Baum–Connes Conjecture was proved for metric spaces which have bounded geometry and embed coarsely into a Hilbert space. If the metric space in question is a finitely generated group, then the coarse Baum–Connes Conjecture implies the Novikov Conjecture for such a group. These applications are discussed at the end of the book. See also [253] for a survey.

Many interesting results on compression were obtained, we refer the reader to [8], [9], [15], [17], [16], [55], [64], [176], [177], [224] and references therein. In particular, some of the ideas used to study compression were used to answer a question of Atiyah about existence of groups with irrational ℓ_2-Betti numbers [14], [13].

The purpose of the uniform Roe algebra is to encode the geometry of the metric space in the spirit of non-commutative geometry. Ideas of non-commutative geometry were developed in large part by Connes and have found remarkable applications in classical geometric settings. We refer to Connes' book [60] for a comprehensive guide to non-commutative geometry.

Chapter 6
Group actions on Banach spaces

In this chapter we will discuss two properties of groups, Kazhdan's property (T) and Gromov's a-T-menability. Neither property is a large scale geometric invariant, however, they are related to geometric properties by various constructions.

6.1 Affine isometric actions

Let E be a normed vector space over $\mathbb{R}$. A *linear isometry* $U: E \to E$ is a linear, one-to-one surjective map which preserves the norm:

$$\|Uv\|_E = \|v\|_E$$

for every $v \in E$. Linear isometries are invertible, the identity operator is an example of a linear isometry. The set of all linear isometries of E forms a group under composition. We will denote this group by $\mathrm{Iso}(E)$.

Let $E = H$ be a Hilbert space. A linear operator U on H is a *unitary operator* if it preserves the scalar product on H,

$$\langle Uv, Uw \rangle = \langle v, w \rangle$$

for every $v, w \in H$. It follows from the parallelogram law that unitary operators are the linear isometries of H. The group of unitary operators on a Hilbert space H is called the unitary group of H and denoted by $U(H)$.

Definition 6.1.1. Let G be a group and E a normed space. An *isometric representation* of G on E is a homomorphism $\pi: G \to \mathrm{Iso}(E)$.

If $E = H$ is a Hilbert space and $\mathrm{Iso}(E) = U(H)$, then we say that π is a unitary representation of G on H.

The following are examples of isometric representations.

Example 6.1.2. The trivial representation on E is defined by setting

$$\pi_g = \mathrm{Id}_E$$

for every $g \in G$.

Example 6.1.3. The regular representation of G on $E = \ell_p(G)$, $1 \leq p \leq \infty$, is defined using the natural action of G on itself by translations and we have already encountered it (though without naming it properly) when discussing amenability. For every $g \in G$ we define an operator $\lambda_g \in \mathrm{Iso}(\ell_p(G))$ by the formula

$$\lambda_g f(h) = g \cdot f(h) = f(hg)$$

for $f \in \ell_p(G)$. It is clear that λ_g is a linear isometry and that λ is a representation. In a similar manner one can define a regular representation of G on $L_p(X, \nu)$ whenever (X, ν) is a measure space on which G acts by measure-preserving transformations.

Affine isometries are maps on E that are a composition of a linear isometry and a translation.

Definition 6.1.4. An *affine isometry* of E is a map $A\colon E \to E$ such that for every $v \in E$ we have

$$Av = Uv + b,$$

where U is a linear isometry and $b \in E$ is a fixed vector.

It turns out that this is the only type of distance-preserving bijection one can have for a normed linear space. By the Mazur–Ulam theorem [161], any surjective, *metric* isometry $f : E \to E$ of a normed space over $\mathbb{R}$ that satisfies only

$$\|f(v) - f(w)\|_E = \|v - w\|_E$$

for all $v, w \in E$ is in fact an affine isometry. (For linear spaces over $\mathbb{C}$ this is not necessarily the case.)

Let now G be a group and E be a normed space. We say that G acts on E by affine isometries if for every $g \in G$ there exists an affine isometry $A_g : E \to E$ such that for any $g, h \in G$ we have

$$A_{gh} = A_g A_h.$$

Since each A_g is of the form described in Definition 6.1.4, the existence of A_g corresponds to the existence of two maps, $\pi : G \to \mathrm{Iso}(E)$ and $b\colon G \to E$, such that

$$A_g v = \pi_g v + b_g. \tag{6.1}$$

Since A_g is an action, we have

$$A_{gh} v = A_g(A_h v) = \pi_g \pi_h v + \pi_g b_h + b_g$$

for every $g, h \in G$. On the other hand,

$$A_{gh}(v) = \pi_{gh} v + b_{gh}$$

for every $g, h \in G$. The comparison of the right-hand sides of the above equalities yields a necessary and sufficient condition for A to define an action:

$$\pi_{gh} = \pi_g \pi_h$$

and

$$b_{gh} = \pi_g b_h + b_g \tag{6.2}$$

for every $g, h \in G$. Condition (6.2) is called the *cocycle condition* and b is said to be a cocycle for π.

Example 6.1.5. Let G be a group and $\pi : G \to \mathrm{Iso}(E)$ be an isometric representation. Fix $v_0 \in E$ and define $b : G \to E$ by

$$b_g = \pi_g v_0 - v_0 \tag{6.3}$$

for every $g \in G$. Then b is a cocycle for π. Thus

$$A_g v = \pi_g v + (\pi_g v_0 - v_0),$$

$v \in E$, defines an affine isometric action of G on E. A cocycle of the form (6.3) will be called a coboundary.

Example 6.1.6. Let $E = \mathbb{R}^n$ and $G = \mathbb{Z}^n$. Then G acts by affine isometries on H. Indeed, consider the action given by

$$A_{(z_1,\ldots,z_n)}(v_1, \ldots, v_n) = (v_1 + z_1, \ldots, v_n + z_n)$$

for $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ and $(v_1, \ldots, v_n) \in \mathbb{R}^n$. In this case the unitary representation is trivial and

$$b_{(z_1,\ldots,z_n)} = (z_1, \ldots, z_n) \in E.$$

6.2 Metrically proper actions and a-T-menability

As mentioned earlier, the usual notion of properness does not make sense in the realm of infinite-dimensional Banach spaces, since they are not locally compact. The version of properness we will use is related to metric properties of the action.

Definition 6.2.1. Let G be a finitely generated group acting by affine isometries on a normed space E, where the action is defined by (6.1). The action is said to be *metrically proper* if

$$\lim_{|g| \to \infty} \|b_g\|_E = \infty.$$

Note that if an action is metrically proper, then for every $v \in E$ we have

$$\lim_{|g| \to \infty} \|A_g v\|_E = \infty,$$

since

$$\|A_g v\|_E = \|\pi_g v + b_g\|_E \geq |\, \|b_g\|_E - \|\pi_g v\|_E \,| \geq |\, \|b_g\|_E - \|v\|_E \,|.$$

If E is a finite-dimensional space, then properness and metric properness of actions coincide.

Definition 6.2.2. A finitely generated group G is called *a-T-menable* if there exists a metrically proper action by affine isometries of G on a Hilbert space.

A-T-menability is also known as the Haagerup property. This property was introduced independently and in different settings by Gromov [108] and Haagerup [120].

Example 6.2.3. The translation action of $\mathbb{Z}^n$ on $\mathbb{R}^n$, constructed in Example 6.1.6, is metrically proper.

In Example 6.1.5 we constructed a cocycle for any representation of any group G. However, cocycles of this form are not useful for proving a-T-menability since they are always bounded:

$$\|\pi_g v_0 - v_0\| \leq \|\pi_g v_0\| + \|v_0\| = 2\|v_0\|$$

for every $g \in G$.

The main reason for introducing a-T-menability in this chapter is that the relation of a-T-menability to coarse embeddability into Hilbert spaces is the same as that of amenability to property A, namely coarse embeddability is a non-equivariant version of a-T-menability.

Proposition 6.2.4. *Let G be a finitely generated group which admits a metrically proper, affine, isometric action on a normed space E, with a cocycle b. Then b is a coarse embedding.*

Proof. We have

$$b_g - b_h = b_{h(h^{-1}g)} - b_h = \pi_h b_{h^{-1}g} + b_h - b_h = \pi_h b_{h^{-1}g}.$$

Hence,

$$\|b_h - b_g\| = \|\pi_h b_{h^{-1}g}\| = \|b_{h^{-1}g}\| \tag{6.4}$$

for every $g, h \in G$. For $t \in \mathbb{R}$ define

$$\rho_-(t) = \inf\{\|b_h\| \mid |h| \geq t\}.$$

Then clearly, $\rho_-(t) \to \infty$ and, by equation (6.4), we have

$$\|b_h - b_g\| = \|b_{h^{-1}g}\| \geq \rho_-(|h^{-1}g|) = \rho_-(d(h, g))$$

for any $g, h \in G$. □

The philosophy outlined above suggests that amenability should imply a-T-menability, just as property A implies coarse embeddability into the Hilbert space. This result is due to Bekka, Cherix and Valette [27]. In fact, their theorem predates its coarse counterpart and was the original motivation behind Theorem 5.2.4.

Theorem 6.2.5. *Let G be a finitely generated amenable group. Then G admits a metrically proper, affine, isometric action on a Hilbert space.*

Proof. By definition of amenability, for every $n \in \mathbb{N}$ there exists a finitely supported $f \in \ell_2(G)$ such that $\|f\|_2 = 1$ and

$$\|f_n - \lambda_g f_n\|_2 = \|f_n - g \cdot f_n\|_2 \leq \frac{1}{2^n}$$

for all $|g| \leq n$, where λ is the left regular representation of G on $\ell_2(G)$, described in Example 6.1.3. Let

$$H = \bigoplus_{n=1}^{\infty} \ell_2(G),$$

and let $\pi : G \to U(H)$ be defined as

$$\pi_g = \bigoplus_{n=1}^{\infty} \lambda_g.$$

Set

$$b_g = \bigoplus_{n=1}^{\infty} (\lambda_g f_n - f_n).$$

We claim that the formula

$$A_g v = \pi_g v + b_g$$

defines an action with the required properties.

We need to first check that b is well defined and that it satisfies the cocycle identity with respect to π. For every $g \in G$ we have

$$\|b_g\|_H^2 = \sum_{n=1}^{\infty} \|\lambda_g f_n - f_n\|_2^2 \leq \sum_{n=1}^{|g|} \|\lambda_g f_n - f_n\|_2^2 + \sum_{n=|g|+1}^{\infty} \frac{1}{4^n} < \infty,$$

which shows that $b_g \in H$ for every $g \in G$. To verify the cocycle condition we write

$$\begin{aligned} \pi_g b_h + b_g &= \pi_g\Big(\bigoplus_{n=1}^{\infty} \lambda_h f_n - f_n\Big) + \Big(\bigoplus_{n=1}^{\infty} \lambda_g f_n - f_n\Big) \\ &= \Big(\bigoplus_{n=1}^{\infty} \lambda_{gh} f_n - \lambda_g f_n + \lambda_g f_n - f_n\Big) = b_{gh}. \end{aligned}$$

It follows that b is well defined and A_g defines an affine isometric action. It remains to show that the action is metrically proper. For every $n \in \mathbb{N}$ there exists a number $N_n \in \mathbb{N}$ such that $|g| \geq N_n$ implies

$$\|\lambda_g f_n - f_n\|_2 = \sqrt{2}$$

since each f_n is finitely supported. Without loss of generality we can assume that N_n increases with n. Let

$$\phi(g) = \max\{n \in \mathbb{N} \mid N_n \leq |g|\}.$$

Then

$$\|b_g\|_H^2 = \sum_{n=1}^{\infty} \|\lambda_g f_n - f_n\|_2^2 \geq \sum_{n=1}^{\phi(g)} 2 = 2\phi(g).$$

The theorem is now proved since $\phi(g) \to \infty$ as $|g| \to \infty$. □

The following diagram describes the relations we have proved so far:

$$\begin{array}{ccc}
\text{amenability} & \Longrightarrow & \text{property A} \\
\Big\Downarrow & & \Big\Downarrow \\
\text{a-T-menability} & \Longrightarrow & \text{coarse embedding into } \ell_2
\end{array}$$

In fact, the reader can observe that in the above diagram the notions on the right are metric, "non-equivariant" versions of the notions on the left. The question whether there are groups which coarsely embed into the Hilbert space but do not have property A is open. The corresponding question in the equivariant setting has an immediate answer: the class of a-T-menable groups is strictly larger than the class of amenable groups.

Proposition 6.2.6. *The free group $\mathbb{F}_n$ is a-T-menable for any $n \geq 2$.*

Proof. Denote by $E_\pm$ the set of oriented edges of the Cayley graph of $\mathbb{F}_2$; every geometric edge linking elements g and h in G in the Cayley graph defines two oriented edges in $E_\pm$: (g, h) and (h, g). Define an edge (h, g) to be positively oriented if h lies between g and the identity element e, and negatively oriented in the opposite situation. The sets of positively and negatively oriented edges will be denoted by E_+ and E_-, respectively. The sign function $\operatorname{sign}(a)$ of an edge $a \in E_\pm$ is defined to be 1 if $a \in E_+$ and -1 if $a \in E_-$. G acts on $E_\pm$ by translation and gives rise to a unitary representation π of G on $\ell_2(E_\pm)$.

For any $g \in \mathbb{F}_n$ consider the function $b\colon \mathbb{F}_n \to \ell_2(E_\pm)$,

$$b_g = \sum \operatorname{sign}(a) 1_a,$$

where a runs through all the edges in $E_\pm$ lying on the unique path linking e and g. Let $g, h \in G$. Then

$$\pi_h b_g = \sum \operatorname{sign}(a) 1_a,$$

where a are the edges in $E_\pm$ lying on the unique path connecting hg and h. Thus,

$$b_{hg} = \pi_h b_g + b_h,$$

or in other words, b is a cocycle for the representation π. Finally, b is metrically proper by similar computations as in Example 5.2.2. □

At this point it is natural to ask whether hyperbolic groups are a-T-menable, however, this is not the case. There exist examples of hyperbolic groups which do not admit metrically proper actions on Hilbert space – in fact, have property (T), which will be discussed in later sections. Hyperbolic groups are however known to act properly by affine isometries on ℓ_p-spaces for $p \geq 2$, where p is sufficiently large and depends on the group [180], [251]. We discuss this case in the next section.

6.3 Actions on ℓ_p-spaces and reflexive Banach spaces

We will now sketch some of the constructions of metrically proper, affine, isometric actions of groups on uniformly convex and reflexive Banach spaces. Below we give a sketch of such a construction for certain class of hyperbolic groups.

Theorem 6.3.1. *Let M be a compact, negatively curved manifold* (*i.e., with sectional curvature* < 0). *Then $\Gamma = \pi_1(M)$ admits a metrically proper affine isometric action on an L_p-space for sufficiently large $p \geq 2$.*

Sketch of proof. Denote by $\widetilde{M}$ the universal covering space of M and by $T\widetilde{M}$ the tangent bundle of $\widetilde{M}$. Consider the Banach space $L_p(\widetilde{M}, T\widetilde{M})$, of L_p-sections of the tangent bundle of $\widetilde{M}$:

$$L_p(\widetilde{M}, T\widetilde{M}) = \Big\{\xi \colon \widetilde{M} \to T\widetilde{M} \mid \int_{\widetilde{M}} \|\xi(x)\|^p_{T_x\widetilde{M}} dV < \infty\Big\}.$$

Here, $T_x\widetilde{M}$ denotes the tangent space at $x \in \widetilde{M}$ and dV denotes the volume form. Let $y \in \widetilde{M}$. For every $x \in \widetilde{M}$ there exists a unique geodesic $\gamma(x, y)$ in $\widetilde{M}$ connecting y and x. Let $\eta_y(x)$ be a unit vector tangent to $\gamma(x, y)$ at x, pointing outward from y. (Note that $\eta_y(y)$ is not defined.) The function $\eta \colon \widetilde{M} \to T\widetilde{M}$ gives a section of the tangent bundle $T\widetilde{M}$, however this section is only bounded and in general not in $L_p(\widetilde{M}, T\widetilde{M})$.

Given a diffeomorphism $f : M \to M$ we will denote by f_* the map induced on the tangent spaces, $f_* \colon T_pM \to T_{f(p)}M$. Let π denote the representation of Γ on sections of the tangent bundle induced by translations, i.e.,

$$(\pi_g\xi)(x) = g_*(\xi(g^{-1}x))$$

for all $g \in G$, $x \in \widetilde{M}$, $\xi \colon \widetilde{M} \to T\widetilde{M}$ and $g_* : T_{g^{-1}x}\widetilde{M} \to T_x\widetilde{M}$. When restricted to $L_p(\widetilde{M}, T\widetilde{M})$, this action is isometric.

Fix $y \in \widetilde{M}$. Define an affine action by setting

$$A_g \xi = \pi_g \xi + b_g,$$

where

$$b_g = \pi_g \eta_y - \eta_y$$

is a cocycle associated to π. The action will be an affine isometric action on $L_p(\widetilde{M}, T\widetilde{M})$, for some $p \geq 2$, once we show that $b_g \in L_p(\widetilde{M}, T\widetilde{M})$ for every $g \in G$. Note that

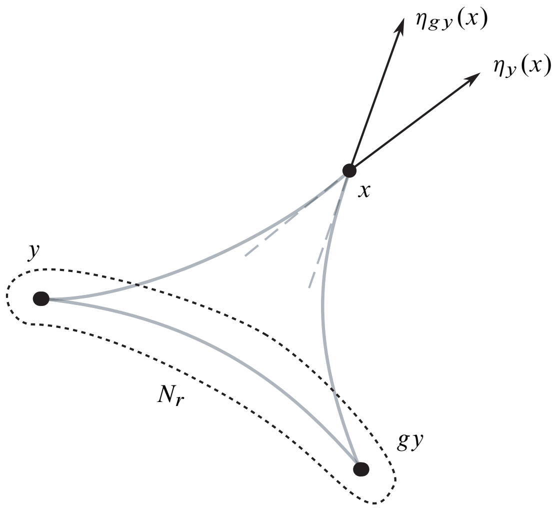

Figure 6.1. The vectors tangent to paths in $\widetilde{M}$.

$$\begin{aligned} \|\pi_g \eta_y - \eta_y\|_p^p &= \int_{\widetilde{M}} \|\eta_y(g^{-1}x) - \eta_y(x)\|_{T_x\widetilde{M}}^p \, dV \\ &= \int_{\widetilde{M}} \|\eta_{gy}(x) - \eta_y(x)\|_{T_x\widetilde{M}}^p \, dV \\ &= \|\eta_{gy} - \eta_y\|_p^p. \end{aligned}$$

Since $\eta_y(x)$ and $\eta_{gy}(x)$ are both unit vectors in $T_x\widetilde{M}$, we can use the standard formula for the dot product, together with the parallelogram law in $T_x\widetilde{M}$, to obtain

$$\|\eta_{gy}(x) - \eta_y(x)\|_{T_x\widetilde{M}} = 2(1 - \cos\theta),$$

where θ is the angle between the geodesics $\gamma(x, gy)$ and $\gamma(x, y)$ measured at x. The angle θ decays exponentially. Using a comparison theorem from differential geometry (see e.g. [54]), we obtain

$$\|\eta_{gy}(x) - \eta_y(x)\|_{T_x \widetilde{M}} \leq C \exp(-kd(x,y))$$

for some $C, k > 0$. We now need to find $p \geq 2$ such that

$$\int_{\widetilde{M}} \exp(-pkd(x,y))\, dV < \infty.$$

However, since the volume function of $\widetilde{M}$ is growing exponentially, there exists $p \geq 2$ sufficiently large to satisfy the above condition. For such p we have $b_g \in L_p(\widetilde{M}, T\widetilde{M})$.

The second goal is to prove that the action on $L_p(T\widetilde{M})$, for the p chosen above, is metrically proper. Let N_r denote the r-neighborhood of the geodesic connecting y and gy. After integrating over N_r we obtain

$$\|\eta_{gy} - \eta_y\|_p \geq \mathrm{Vol}(N_r)$$

for some $r > 0$ independent of g and y, and the latter tends to infinity as $|g| \to \infty$. □

The above argument can be generalized to the discrete setting, however the existence of tangent vectors has to be adequately compensated for. The following generalization was proved in [251].

Theorem 6.3.2. *Let G be a finitely generated hyperbolic group. There exists $p_0 \geq 2$ such that G admits a metrically proper affine isometric action on the space $\ell_p(G \times G)$ for all $p \geq p_0$.*

A more recent result of B. Nica [180] shows that every hyperbolic group acts properly by affine isometries on $L_p(\partial G \times \partial G)$, where ∂G denotes the boundary of the group and $p \geq 2$ is sufficiently large. In addition, Nica gives a lower estimate on such a p in terms of a certain type of dimension of the boundary.

Problem 6.3.3. Let $p_0 > 2$. Construct a finitely generated group G such that G acts properly by affine isometries on $L_p(\nu)$ for $p \geq p_0$ but not on $L_p(\nu)$ for $p < p_0$.

Just as in the case of coarse embeddings, it is natural to expect that once we weaken the requirements on the Banach space, more groups will act on it properly. In particular, Theorem 5.4.3 has its equivariant version.

Theorem 6.3.4 ([45]). *Let G be a finitely generated group. There exists a reflexive Banach space E on which G acts properly by affine isometries.*

More precisely, there exists an increasing sequence of numbers p_n, satisfying $\lim_{n\to\infty} p_n = \infty$, such that G acts properly by affine isometries on the ℓ_2-direct sum $E = (\bigoplus_{n=1}^{\infty} \ell_{p_n}(G))_{(2)}$. The linear part of this affine action is the direct sum of the left regular representations of G on $\ell_{p_n}(G)$.

To prove this fact we need an equivariant version of 5.4.4.

Lemma 6.3.5. *Let G be a finitely generated group. Then for every $n \in \mathbb{N}$ there exists a function $f : G \to \mathbb{R}$, such that*

(1) $\|f\|_\infty = 1$,
(2) $\|f - s \cdot f\|_\infty \leq \frac{1}{n}$ *for every generator s of G, and*
(3) $\operatorname{supp} f \subseteq B(e, n)$.

The proof of Theorem 6.3.4 amounts to noticing that for a discrete group the construction in 5.4.4 is in fact equivariant. Theorem 6.3.4 follows from the above lemma by the cocycle construction that we have used several times before. We leave the proof as an exercise.

We will now consider a class of representation Hilbert spaces which is larger than the class of unitary representations.

Definition 6.3.6. Let E be a Banach space and let G be a group. A *uniformly bounded representation* of G on E is a homomorphism $\pi : G \to B(E)$ into the algebra of bounded operators[1] on E such that

$$\|\pi\| = \sup_{g \in G} \|\pi_g\| < \infty.$$

A uniformly bounded representation of a group G on a Banach space E can be viewed as an isometric representation of G on a space E' isomorphic to E. Indeed, the formula

$$\|v\|' = \sup_{g \in G} \|\pi_g v\|_E$$

defines a new norm on E, which is equivalent to $\|\cdot\|_E$ (i.e., the identity map on the underlying vector space is an isomorphism). Uniformly bounded representations play an important role in abstract harmonic analysis.

Open question 6.3.7 (Y. Shalom). *Is it true that for every hyperbolic group G there exists a metrically proper, affine action A of G on a Hilbert space such that the linear part of A is a uniformly bounded representation of G on H?*

6.4 Kazhdan's property (T)

In this section we introduce property (T), which can be vaguely described as a property opposite to a-T-menability. In Chapter 3 we proved the Hulanicki–Reiter condition, characterizing amenability in terms of ℓ_1-norms of the finitely supported functions $f - s \cdot f$ for generators $s \in S$. The same characterization remains true if

[1] More precisely, the homomorphism is into the group of invertible, bounded operators on E.

we use ℓ_2-norms, and we use it to formulate non-amenability: *a finitely generated group G is not amenable if and only if there exists a constant $c > 0$ such that*

$$c\|f\|_2 \leq \|f - s \cdot f\|_2$$

for every finitely supported function $f: G \to \mathbb{R}$ and some generator s of G. We have the following property of the regular representation.

Proposition 6.4.1. *The regular representation of an infinite group G on $\ell_2(G)$ does not have any non-zero invariant vectors. More precisely, there does not exist a non-zero vector $v \in \ell_2(G)$ such that $g \cdot v = v$ for every $g \in G$.*

A group satisfies property (T) if all unitary representations display a similar behavior. We will say that a representation π of a finitely generated group G on a Banach space E has almost invariant vectors if for every $\varepsilon > 0$ there exists a vector $v \in E$, $\|v\|_E = 1$, such that

$$\|\pi_s v - v\|_E \leq \varepsilon$$

for every generator $s \in G$.

Definition 6.4.2. A finitely generated group has *property* (T) if every unitary representation which has almost invariant vectors has a non-zero invariant vector.

Example 6.4.3. Finite groups have property (T). Let G be such a group and we choose the entire group G as the generating set. Let $\pi: G \to U(H)$ be a unitary representation which has almost invariant vectors; that is, for every $\varepsilon > 0$ there exists a non-zero vector $v \in H$ which satisfies $\|v\| = 1$ and

$$\sup_{g \in G} \|\pi_g v - v\| \leq \varepsilon$$

for all $g \in G$. Take the orbit

$$Gv = \{\pi_g v\}_{g \in G}$$

of v in H and its closed convex hull, denoted by $\operatorname{conv}(Gv)$. This set is invariant under the action of the representation π,

$$\pi_g(\operatorname{conv}(Gv)) = \operatorname{conv}(Gv),$$

because the orbit Gv is. Since v is a non-zero vector, choosing a sufficiently small ε, we obtain an orbit Gv such that $\operatorname{conv}(Gv)$ does not contain 0. Recall that given a closed convex set K in a Hilbert space, there is always a unique vector w such that

$$\|w\|_H = d(0, K) = \inf\{\|x\|_H \mid x \in K\}.$$

Let w be the vector realizing the distance of Gv from the origin. Then w is a non-zero invariant vector of π.

More generally, for an infinite amenable group G, the regular representation on $\ell_2(G)$ has almost invariant vectors, but no non-zero invariant vectors. Thus infinite amenable groups do not have property (T).

Explicit examples of infinite groups with property (T) are difficult to find. A classical example is furnished by certain linear groups.

Theorem 6.4.4. *The group* $\mathrm{SL}(n, \mathbb{Z})$ *has property* (T) *for* $n \geq 3$.

The proof of the above theorem is quite involved and we will not present it here. Instead we refer the reader to [26], [152] for proofs of the above fact. In the following sections we will provide a sufficient condition for property (T), based on spectral properties of the discrete Laplacian. This condition allows one to produce other examples of groups with property (T).

It is easy to see that property (T) is not, in general, inherited by subgroups, since groups such as $\mathrm{SL}_3(\mathbb{Z})$ have infinite cyclic subgroups which are amenable. We will prove in section 6.6 that property (T) is preserved under epimorphisms. As for geometric invariance, it was a long-standing open problem whether property (T) is a quasi-isometry invariant, however this is not the case. The example, attributed to Gersten, can be found in [26].

We have discussed a parallel between the "equivariant" and coarse versions of certain notions for groups. In [209] Roe posed the following question.

Open question 6.4.5. *How should we define coarse property* (T)?

As explained in [209], coarse property (T) should be a coarse invariant, should contradict the existence of a coarse embedding into a Hilbert space, and for groups it should imply property (T). A notion of geometric property (T) was introduced by Willett and Yu in [240], [241] in their study of the K-theory of the maximal Roe algebra. For a coarse disjoint union of graphs, the geometric property (T) implies the property of being an expander. In particular, the coarse disjoint union of finite quotients $\coprod_i G/N_i$, $\bigcap N_i = \{e\}$, has geometric property (T) if and only if G has property (T). However, for infinite groups this geometric property (T) of Willett and Yu is equivalent to non-amenability and thus is weaker than property (T).

6.5 Fixed points and Kazhdan's property (T)

Property (T) is closely related to the existence of fixed points for affine isometric actions. A fixed point for an affine action A of a group G on a linear space E is a vector $v \in E$ such that

$$A_g v = v$$

for all $g \in G$. If G is generated by a finite set S, then v is a fixed point of the action if and only if

$$A_s v = v$$

for every $s \in S$. Cocycles of the form $\pi_g w - w$ are of special importance when studying fixed points.

Proposition 6.5.1. *Let A be an affine, isometric action with linear part π of a finitely generated group G on a normed space E. The vector $w \in E$ is a fixed point of A if and only if the cocycle b associated to this action satisfies*

$$b_g = w - \pi_g w \tag{6.5}$$

for every $g \in G$.

Proof. If the cocycle for the action is of the form (6.5), then

$$A_g v = \pi_g v + (w - \pi_g w) = \pi_g (v - w) + w.$$

Letting $v = w$ we see that

$$A_g w = w$$

for every $g \in G$.

Conversely, assume that an affine isometric action A has a fixed point w and let b be the cocycle associated to A. Then

$$b_g = A_g w - \pi_g w = w - \pi_g w$$

for every $g \in G$. Hence the cocycle is of the required form. □

The next theorem is due to Guichardet [119].

Theorem 6.5.2. *Let G be a finitely generated group. G has property (T) if every affine isometric action of G on a Hilbert space has a fixed point.*

Proof. Assume the contrary, namely that G does not have property (T). Then there exists a unitary representation π on a Hilbert space H without invariant vectors and with the property that for every $n \in \mathbb{N}$ there exists a unit vector $v_n \in H$ satisfying

$$\sup_{g \in G} \|\pi_g v_n - v_n\| \leq 2^{-n}$$

whenever $|g| \leq n$. Now consider the Hilbert space $\mathcal{H} = \bigoplus_{n=1}^{\infty} H$ with the representation $\bar{\pi} = \bigoplus_{n=1}^{\infty} \pi$. Define a cocycle $b\colon G \to \mathcal{H}$ by the formula

$$b_g = \bigoplus_{n=1}^{\infty} n(\pi_g v_n - v_n).$$

It is easy to check that b is well defined. We will show that b is not bounded. By the fact that π does not have invariant vectors, for each $n \in \mathbb{N}$ there exists $g_n \in G$ such that

$$\|\pi_{g_n} v_n - v_n\| \geq 1.$$

Indeed, an argument similar to the one in Example 6.4.3 would otherwise produce an invariant vector. Therefore,

$$\|b_{g_n}\| \geq n\|\pi_{g_n} v_n - v_n\| \geq n. \qquad \square$$

Note that the above shows that the difference between a group without property (T) and an a-T-menable group is the difference between existence of unbounded and metrically proper cocycles. There are groups which do not have property (T) and are not a-T-menable, for instance a direct product of an infinite group with property (T) and an infinite a-T-menable group has this property.

Theorem 6.5.2 has a converse, turning existence of fixed points for every affine, isometric action into a characterization of property (T).

6.6 Construction of expanders

Recall that a coarse disjoint union of expanders is an example of a metric space, which does not embed coarsely into the Hilbert space. We will now use Kazhdan's property (T) to construct examples of expanders. This construction, due to Margulis [156], was the first explicit construction of such graphs.

We first define the notion of a Kazhdan constant.

Definition 6.6.1. Let G be a group with property (T) generated by a finite set S. Define the *Kazhdan constant* $\kappa_{(G,S)}$, associated to G and S, to be

$$\kappa_{(G,S)} = \inf_{\pi} \inf_{\|v\|=1} \sup_{s \in S} \|\pi_s v - v\|,$$

where the first infimum is taken over all unitary representations π without invariant vectors, and the second infimum is taken over all unit vectors in the Hilbert space on which the representation π is defined.

We leave it as an exercise that a group has property (T) if and only if it has a positive Kazhdan constant. Kazhdan constants are a way of quantifying property (T). They also have a close connection to the first non-zero eigenvalue of the Laplacian used to describe expanders. Given a group G generated by a finite set S and a surjection $\varphi\colon G \to \Gamma$, the set $\varphi(S)$ is a generating set for Γ.

Theorem 6.6.2. *Let G and Γ be finitely generated groups and let $\varphi\colon G \to \Gamma$ be a surjective homomorphism. Then*

$$\kappa_{(G,S)} \leq \kappa_{(\Gamma,\varphi(S))}.$$

In particular, if G has property (T) then Γ has property (T) as well.

Proof. Let S be a generating set of G and let $\pi : \Gamma \to U(H)$ be a unitary representation of Γ on some Hilbert space H. Assume that π does not have non-zero invariant vectors. We pull π back to a representation $\bar{\pi} : G \to U(H)$ by the formula

$$\bar{\pi}_g = \pi_{\varphi(g)}$$

for every $g \in G$. The representation $\bar{\pi}$ does not have non-zero invariant vectors and, since G has (T), we have

$$\|v\| \leq \frac{1}{\kappa_{(G,S)}} \sup_{s \in S} \|\bar{\pi}_s v - v\|$$

for any vector $v \in H$. This means

$$\|v\| \leq \frac{1}{\kappa_{(G,S)}} \sup_{s \in S} \|\pi_{\varphi(s)} v - v\|,$$

which proves the claim. □

Theorem 6.6.3. *Let G be an infinite residually finite group, generated by a finite set S and satisfying Kazhdan's property (T). Let $\{N_i\}_{i=1}^{\infty}$ be a sequence of normal subgroups of G, such that each N_i is of finite index in G and $\bigcap N_i = \{e\}$. Then the collection of finite quotients $\{\coprod G/N_i\}_{i \in \mathbb{N}}$ is a sequence of expanders.*

Proof. For a normal, finite index subgroup N of G consider the space

$$\ell_2^0(G/N) = \{f : G/N \to \mathbb{R} \mid \textstyle\sum_{x \in G/N} f(x) = 0\}.$$

It is a finite-dimensional Hilbert space with respect to the standard inner product

$$\langle f, f' \rangle = \sum_{x \in G/N} f(x) f'(x),$$

and G/N acts on it via the regular representation. Note that the regular representation on a finite group fixes only the constant vectors, however, such vectors (other than 0) do not belong to $\ell_2^0(G/N)$. In other words, this representation does not have non-zero invariant vectors. By the previous theorem,

$$\kappa_{(G,S)} \leq \kappa_{(G/N, \varphi(S))},$$

where $\varphi \colon G \to G/N$ is the quotient map.

Consider a function $f \in \ell_2^0(G/N)$ satisfying $\sum |f(x)|^2 = 1$. Then

$$\begin{aligned}
\sum_{x \in G/N} \sum_{y \in B(x,1)} |f(x) - f(y)|^2 &= \sum_{s \in S} \sum_{x \in G/N} |f(x) - f(xs)|^2 \\
&= \sum_{s \in S} \|f - s \cdot f\|^2 \\
&\geq \sup_{s \in S} \|f - s \cdot f\|^2 \\
&\geq \kappa_{(G,S)}.
\end{aligned}$$

Thus $\lambda_1(G/N) \geq \kappa_{(G,S)}$. Applying the above calculations to $N = N_i$ for every $i \in \mathbb{N}$, we obtain that the sequence of Cayley graphs G/N_i is a sequence of expanders since $\kappa(G,S)$ gives a lower bound on $\lambda_1(G/N_i)$, independent of $i \in \mathbb{N}$. □

Property (T) is a sufficient assumption in the above theorem, however a property weaker than (T) turns the above statement into an equivalence. More precisely, the fact that the finite quotients G/N_i constitute a sequence of expanders can be characterized by a representation theoretic property, a weak version of property (T). This property, known as *property* (τ), was introduced by Lubotzky [152], and we refer to [152], [153] for details.

There is one more consequence of Theorem 6.6.2 that we would like to mention. Since infinite amenable groups do not have property (T), Theorem 6.6.2 can serve as a condition showing that a group does not have (T). Recall that the *commutator subgroup* $[G,G]$ of a group G is the subgroup of G generated by all commutators

$$\{g^{-1}h^{-1}gh \mid g,h \in G\}.$$

The commutator subgroup is the smallest normal subgroup such that the quotient $G/[G,G]$ is abelian. The quotient group $G/[G,G]$ is called the *abelianization* of G, and, intuitively, the larger the abelianization of G, the closer G is to being abelian. The following is a direct consequence of Theorem 6.6.2.

Corollary 6.6.4. *If a finitely generated group G admits a surjective homomorphism onto an infinite amenable group, then G does not have property (T). In particular, a group with property (T) has finite abelianization.*

Example 6.6.5. Any finitely presented group, which has more generators than relations has an infinite abelianization and does not have property (T).

6.7 Property (T) and spectral conditions

We have mentioned without proof examples of groups which have property (T). Proving that they satisfy property (T) is highly non-trivial. In this section we will give a sufficient condition, which relates the eigenvalues of the Laplacian on a certain finite graph, constructed using the generating set, and property (T).

Definition 6.7.1. Let G be a finitely generated group and S be a finite, symmetric generating set, not containing the identity element. We define the *link graph*, $\Gamma_S = (V,E)$, of S as follows:

(1) the vertices are the elements of S,

(2) two vertices $s,t \in S$ are connected by an edge if $s^{-1}t \in S$ (equivalently, $t^{-1}s \in S$).

This graph Γ_S is finite, and if it is additionally connected, the number

$$\lambda_S = \inf \left\{ \frac{\sum_{s\in S}\sum_{t\sim s} |f(s)-f(t)|^2}{\sum_{s\in S} f(s)^2 \deg(s)} \mid f: V \to \mathbb{R},\ \sum_{s\in S} f(s) = 0 \right\}, \tag{6.6}$$

where $t \sim s$ means that t and s are connected by an edge, is positive. Note that the above quantity is, essentially, a generalization of Definition 5.6.1 to a non-regular graph. If the degree of a graph is constant, then inequality (6.6) reduces to the one in Definition 5.6.1.

Our goal in this section is to prove the following fact.

Theorem 6.7.2 ([255]). *Let G be a group generated by a finite symmetric generating set S, not containing the identity element. If the link graph of S is connected and*

$$\lambda_S > \frac{1}{2},$$

then G has Kazhdan's property (T).

We will use the natural duality in Hilbert spaces to prove Theorem 6.7.2. Let

$$T = \{(s,t) \in S \times S \mid s^{-1}t \in S\}.$$

Note that the set T is the set of directed edges of the link graph, in the sense that each geometric edge appears in T twice, once for each of its two possible orientations. Let H denote a Hilbert space with a scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_H$ and with a unitary representation $\pi : G \to U(H)$. Consider the following three Hilbert spaces:

- the space
$$C^0(G;\pi) = H,$$
with the scalar product
$$\langle v, w\rangle_{(0)} = \#T\langle v, w\rangle_H;$$
- the space
$$C^1(G;\pi) = \{f : S \to H \mid f(s^{-1}) = -\pi_{s^{-1}} f(s)\},$$
with the scalar product
$$\langle f, f'\rangle_{(1)} = \sum_{s\in S} \langle f(s), f'(s)\rangle_H \deg(s);$$
- the space
$$C^2(G;\pi) = \{\psi : T \to H\},$$
with the scalar product
$$\langle \psi, \psi'\rangle_{(2)} = \sum_{(s,t)\in T} \langle \psi(s,t), \psi'(s,t)\rangle_H .$$

The associated norms will be denoted by $\|\cdot\|_{(i)}$, for $i = 0, 1, 2$, respectively. We also consider the following operators between the above Hilbert spaces:

- the linear operator $d: C^0(G;\pi) \to C^1(G;\pi)$ defined by

$$dv(s) = \pi_s v - v$$

  for every $v \in C^0(G;\pi)$;
- the linear operator $\delta: C^1(G;\pi) \to C^2(G;\pi)$ defined by

$$\delta f(s,t) = f(s) - f(t) + \pi_s f(s^{-1}t)$$

  for every $f \in C^1(G;\pi)$.

Norm inequalities between these operators will be our main tool in proving Theorem 6.7.2.

Lemma 6.7.3. *d and δ are bounded.*

Lemma 6.7.4. $\delta \circ d = 0$.

The proofs of the above two lemmas are easy exercises. There exists a bounded operator $d^*\colon \ker\delta \to C^0(G;\pi)$ which is adjoint to d, namely it satisfies

$$\langle dv, f\rangle_{(1)} = \langle v, d^* f\rangle_{(0)}$$

for every $v \in C^0(G;\pi)$ and $f \in \ker\delta$. To relate the operator d to property (T) as in Definition 6.4.2 note that v is an invariant vector for π if and only $dv = 0$. The requirement

$$\kappa\|v\|_H \le \|\pi_s v - v\|_H$$

in Definition 6.4.2 can be related to properties of the operators d, d^* and δ.

Proposition 6.7.5. *Let π be a unitary representation of G with no non-zero invariant vectors. Consider the following statements:*

(1) *There exists a constant $\kappa > 0$ such that*

$$\kappa\langle f, f\rangle_{(1)} \le \langle d^* f, d^* f\rangle_{(0)}$$

for every $f \in \ker\delta$.

(2) *The operator d is a surjection onto* $\ker\delta$.

(3) *There exists a constant $\kappa > 0$ such that*

$$\kappa\langle v, v\rangle_{(0)} \le \langle dv, dv\rangle_{(1)}$$

for all $v \in C^0(G;\pi)$.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

If there exists $\kappa > 0$ *such that* (3) *holds for every unitary representation* π, *then* G *has property* (T).

Proof. To prove that (3) implies property (T), note that the inequality $\kappa\langle v, v\rangle_{(0)} \leq \langle dv, dv\rangle_{(1)}$, translated into the inner product on H, gives

$$\kappa\langle v, v\rangle_H \leq \sum_{s\in S} \frac{\deg(s)}{\#T}\langle dv(s), dv(s)\rangle_H$$

for every $v \in H$. Since $\sum_{s\in S} \deg(s) = \#T$, we conclude that for every $v \in H$ there must exist $s \in S$ such that

$$\kappa\langle v, v\rangle_H \leq \langle dv(s), dv(s)\rangle_H.$$

This implies

$$\sqrt{\kappa}\|v\|_H \leq \sup_{s\in S} \|\pi_s v - v\|_H$$

for every $v \in H$.

The fact that (2) $\Longrightarrow$ (3) follows from the Open Mapping Theorem. Indeed, since π does not have invariant vectors, the operator d is injective. If d is additionally surjective then, by the Open Mapping Theorem, there exists a constant $\kappa > 0$ such that

$$\kappa\langle v, v\rangle_{(0)} \leq \langle dv, dv\rangle_{(1)}$$

for every $v \in C^0(G;\pi)$.

To prove the remaining implication, note that if the inequality

$$\kappa\langle f, f\rangle_{(1)} \leq \langle d^* f, d^* f\rangle_{(0)}$$

is satisfied for some $\kappa > 0$, then, by standard duality arguments, the adjoint of d^*, in this case $d = (d^*)^*$, is surjective. □

Thus if we can show that $\langle d^* f, d^* f\rangle_{(0)} \geq C\langle f, f\rangle_{(1)}$ for every unitary representation π without non-zero invariant vectors, then Theorem 6.7.2 will be proved. We start by proving an explicit formula for the operator d^*.

Proposition 6.7.6. *For any* $f \in C^1(G;\pi)$ *we have*

$$d^* f = -2\sum_{s\in S} f(s)\frac{\deg(s)}{\#T}.$$

Proof. We have

$$\begin{aligned}
\langle dv, f\rangle_{(1)} &= \sum_{s\in S}\langle \pi_s v - v, f(s)\rangle_H \deg(s)\\
&= \sum_{s\in S}(\langle \pi_s v, f(s)\rangle_H - \langle v, f(s)\rangle_H)\deg(s)\\
&= \sum_{s\in S}(\langle v, \pi_{s^{-1}} f(s)\rangle_H - \langle v, f(s)\rangle_H)\deg(s).
\end{aligned}$$

Since f satisfies the relation $\pi_{s^{-1}} f(s) = -f(s^{-1})$, we have

$$\begin{aligned}
\langle dv, f\rangle_{(1)} &= \sum_{s\in S}(\langle v, -f(s^{-1})\rangle_H - \langle v, f(s)\rangle_H)\deg(s)\\
&= -\sum_{s\in S}\langle v, f(s)\rangle_H \deg(s) - \sum_{s\in S}\langle v, f(s)\rangle_H \deg(s)\\
&= -2\sum_{s\in S}\langle v, f(s)\rangle_H \deg(s)\\
&= \Big\langle v, -2\sum_{s\in S} f(s)\frac{\deg(s)}{\#T}\Big\rangle_{(0)}.
\end{aligned}$$

Since the above equality holds for every $v \in C^0(G;\pi)$ and $f \in C^1(G,\pi)$, the claim follows. $\square$

We now define a linear operator, $D\colon C^1(G;\pi) \to C^2(G,\pi)$, by the formula

$$Df(s,t) = f(s) - f(t).$$

Then $Df(s,t) = \delta f(s,t) - \pi_s f(s^{-1}t)$. In particular, for $f \in \ker\delta$ we have $Df(s,t) = -\pi_s f(s^{-1}t)$.

Lemma 6.7.7. *For every $f \in \ker\delta$ we have $\|f\|_{(1)} = \|Df\|_{(2)}$.*

Proof. We have

$$\begin{aligned}
\langle Df, Df\rangle_{(2)} &= \sum_{(s,t)\in T}\langle f(s) - f(t), f(s) - f(t)\rangle_H\\
&= \sum_{(s,t)\in T}\langle \pi_s f(s^{-1}t), \pi_s f(s^{-1}t)\rangle_H\\
&= \sum_{(s,t)\in T}\langle f(s^{-1}t), f(s^{-1}t)\rangle_H
\end{aligned}$$

$$= \sum_{\alpha \in S} \sum_{\{(s,t) \mid s^{-1}t = \alpha\}} \langle f(s^{-1}t), f(s^{-1}t) \rangle_H$$
$$= \sum_{s \in S} \langle f(s), f(s) \rangle_H \deg(s)$$
$$= \langle f, f \rangle_{(1)}. \qquad \square$$

Define $M \colon (\bigoplus_{s \in S} H)_{(2)} \to (\bigoplus_{s \in S} H)_{(2)}$ by

$$Mf(s) = -\frac{d^* f}{2}$$

for every $s \in S$. Note that Mf is simply the projection onto the constant functions $f : S \to H$.

Lemma 6.7.8. *For $f \in C^1(G; \pi)$, $\|Mf\|_{(1)} = \frac{1}{2}\|d^* f\|_{(0)}$.*

Proof. We have

$$\|Mf\|_{(1)}^2 = \frac{1}{4} \sum_{s \in S} \|d^* f\|_H^2 \deg(s) = \frac{1}{4}\|d^* f\|_H^2 \# T = \frac{1}{4}\|d^* f\|_0^2. \qquad \square$$

The operator D is essentially the discrete gradient. Therefore, the lower bound on its norm is related to λ_S.

Lemma 6.7.9. *We have*

$$\|Df\|_{(2)} \geq \sqrt{2\lambda_S}\|f - Mf\|_{(1)}$$

for every $f \in (\bigoplus_{s \in S} H)_{(2)}$.

Proof. We note that

$$\|Df\|_{(2)}^2 = 2\Big(\sum_{s \in S} \sum_{t \sim s} \|f(s) - f(t)\|_H^2\Big).$$

Expression (6.6), defining λ_S, can be rephrased as follows: λ_S is the infimum of all $\lambda > 0$ such that the inequality

$$\lambda\Big(\sum_{s \in S} |f(s) - Mf|^2 \deg(s)\Big) \leq \sum_{s \in S} \sum_{t \sim s} |f(s) - f(t)|^2,$$

holds for all $f : V \to \mathbb{R}$. Indeed, recall that $f - Mf$ can be viewed as the orthogonal projection onto the complement of the constant functions on V. Integrating the second inequality and combining the result with the first expression we obtain the claim. $\square$

Theorem 6.7.2 is now a consequence of Proposition 6.7.5 and the following fact.

Proposition 6.7.10. *The inequality*

$$\|d^* f\|_{(0)} \geq 2\Big(1 - \frac{1}{\sqrt{2\lambda_S}}\Big)\|f\|_{(1)}$$

holds for every $f \in \ker\delta$.

Proof. Let $f \in \ker\delta$. By Lemma 6.7.9 we have

$$\begin{aligned}\|Df\|_{(2)} &= \|D(f - Mf)\|_{(2)}\\ &\geq \sqrt{2\lambda_S}\|f - Mf\|_{(1)}\\ &\geq \sqrt{2\lambda_S}\|f\|_{(1)} - \sqrt{2\lambda_S}\|Mf\|_{(1)}.\end{aligned}$$

By Lemmas 6.7.7 and 6.7.8 we have

$$\|f\|_{(1)} \geq \sqrt{2\lambda_S}\|f\|_{(1)} - \frac{\sqrt{2\lambda_S}}{2}\|d^* f\|_{(0)}.$$

Rearranging the terms gives the claim. □

Proof of Theorem 6.7.2. If $\lambda_S > \frac{1}{2}$, then d^* satisfies the conditions of Proposition 6.7.5 and G has property (T). □

A natural question arises how to construct a group with a finite generating set S, whose link graph satisfies $\lambda_S > \frac{1}{2}$. Examples of such groups are given in [255]. These groups are the so-called $\widetilde{A}_2$ *groups* and the graph Γ_S for their generating sets can be represented as the incidence graph of a projective space over a finite field. For such incidence graphs the spectrum of the Laplacian and, in particular, the first non-zero eigenvalue were computed by Feit and Higman [94] and it follows from their computation that the assumptions of Theorem 6.7.2 are satisfied. The fact that these groups have property (T) was first proved in [53] by a different method.

Another class of examples is given by certain random groups, which, with high probability, are hyperbolic and have property (T) [255]. A detailed proof can be found in [148].

Exercises

Exercise 6.1. Prove that the regular representation of any infinite group G on $\ell_2(G)$ does not have non-zero invariant vectors (Proposition 6.4.1).

Exercise 6.2. A dual space of a Banach space E is the space of all linear operators from E to $\mathbb{R}$, with the operator norm. Let π be an isometric representation of a group G on a Banach space E. Show that the contragradient $\bar{\pi}_g = \pi^*_{g^{-1}}$ is an isometric representation of G on E^*.

Exercise 6.3. Prove Lemma 6.3.5. (Hint: use the construction from the proof of Lemma 5.4.4).

Exercise 6.4. Prove Theorem 6.3.4.

Exercise 6.5. Let G be a finitely generated group and let Γ be a finite index subgroup. Show that G has property (T) if and only if Γ has property (T).

Exercise 6.6. Show that if G acts on $\mathcal{H}_1$ properly by affine isometries and G acts on $\mathcal{H}_2$ by affine isometries (not necessarily properly), then there is a direct sum of the actions and this new action is metrically proper.

Exercise 6.7. Let π and ρ be two equivalent representations of G on a Banach space E; that is, there exists an invertible operator T such that $\pi_g = T^{-1}\rho_g T$. Show that every affine π-action has a fixed point if and only if every affine ρ-action has a fixed point.

Exercise 6.8. Prove Lemma 6.7.3.

Exercise 6.9. Prove Lemma 6.7.4.

Exercise 6.10. Show that $f - Mf$ is orthogonal to Mf, with M as in Lemma 6.7.8. Use this and the Pythagorean theorem to improve the estimate in Proposition 6.7.10.

Exercise 6.11. Let π be a unitary representation on H with no invariant vectors. Show that under the assumptions of Theorem 6.7.2 the operator d^* is surjective.

Notes and remarks

The result of Bekka, Cherix and Valette [27] that amenability implies a-T-menability provided the inspiration for Theorem 5.2.4, proved in [250].

A-T-menability is of interest as a weak amenability-type property. For instance, Farley [87] proved that Thompson's group F is a-T-menable. Applications of a-T-menability include, via a result of Higson and Kasparov [126], the Baum–Connes conjecture with arbitrary coefficients. The class of a-T-menable groups includes certain groups acting on ultrametric spaces [133] and Baumslag–Solitar groups [96].

Kazhdan's property (T) was introduced in [145] and since then has found numerous application in various fields, such as dynamical systems and ergodic theory, functional analysis and von Neumann algebras. Originally Kazhdan used it to prove a conjecture of Siegel that lattices in semi-simple Lie groups are finitely generated. Apart from a construction of expanders another spectacular application of property (T) was the solution of the Ruziewicz problem, obtained by Margulis [157] and Sullivan [226]. Using the work of Rosenblatt [213] they showed that the Lebesque measure is the only rotation-invariant mean on S^n for $n \geq 4$. The same problem for $n = 2, 3$ was solved by Drinfeld [80].

The argument in Section 6.7 is part of the geometric approach to property (T). It is based on the work of Garland [97]. Other results for property (T), following a similar strategy, include [19], [81], [85], [144], [199], [254]. More generally, spectral conditions also provide sufficient conditions for the existence of fixed points on non-positively curved spaces; see [135], [234], [235]. A systematic study of generalizations of property (T) to other Banach spaces was done in [18]. A generalization of the spectral method to other Banach spaces was obtained in [186].

Chapter 7
Coarse homology

In this chapter we will explore some algebraic functors in coarse geometry, namely various versions of coarse homology groups. Coarse homology and cohomology was introduced by Roe in [207]. A uniformly finite (coarse) homology was introduced by Block and Weinberger in [34]. The main motivation was index theory, however coarse homology proved itself to be of independent interest. Basic knowledge of simplicial homology and algebraic topology is assumed in this section, see e.g. [124], [175].

7.1 Coarse locally finite homology

We start with a definition of the chain complex. Recall that an n-simplex, denoted by $\bar{x}$, is an $(n+1)$-tuple

$$\bar{x} = [x_0, \dots, x_n],$$

where $x_i \in X$. Thus an n-simplex is an element of X^{n+1}. We equip the Cartesian product X^{n+1} with the sup metric:

$$d_{X^{n+1}}((x_0, x_1, \dots, x_n), (y_0, y_1, \dots, y_n)) = \max_{i=0,\dots,n} d_X(x_i, y_i).$$

Chain groups are then linear spaces spanned by certain linear combinations on simplices.

Definition 7.1.1. Let X be a uniformly discrete, bounded geometry metric space and let $\mathcal{R}$ be a commutative group. Define the *coarse n-chain group*, $CX_n(X, \mathcal{R})$, to be the linear space of formal linear combinations,

$$c = \sum_n c_{\bar{x}} \bar{x},$$

where $c_{\bar{x}} \in \mathcal{R}$ and $\bar{x} = [x_0, \dots, x_n]$ such that for each chain c there exists a number $P_c > 0$, called the propagation of c, with $c_{\bar{x}} = 0$, provided that $\max_{i,j} d(x_i, x_j) \geq P_c$.

Of course we require that chains c in the above definition are indeed chains in the usual sense; that is, $c_{-\bar{x}} = -c_{\bar{x}}$, where $-\bar{x}$ denotes a simplex with the orientation opposite to $\bar{x}$. The propagation condition in the above definition states that for any given chain c there is a number P_c such that simplices of diameter larger than P_c

do not appear in c. Yet another way to phrase this requirement is to say that every n-chain is supported on a bounded neighborhood of the $(n+1)$-diagonal in X^{n+1}.

We consider the standard differential, $\partial_n \colon CX_n(X,\mathcal{R}) \to CX_{n-1}(X,\mathcal{R})$, defined on simplices by the formula

$$\partial_n([x_0,\dots,x_n]) = \sum_{i=0}^{n}(-1)^i[x_0,\dots,\hat{x_i},\dots,x_n],$$

where the hat denotes omission of the term. We extend ∂_n to the whole $CX_n(X,\mathcal{R})$ by linearity. The proof of the following is standard.

Proposition 7.1.2. *For every $n \in \mathbb{N}$, $\partial_n \circ \partial_{n+1} = 0$*

The above proposition allows us to define coarse locally finite homology groups.

Definition 7.1.3. The *coarse* (*locally finite*) *homology groups* of a uniformly discrete, bounded geometry metric space X are defined by

$$\begin{aligned} &HX_n(X,\mathcal{R}) \\ &\quad = \ker\partial\colon CX_n(X,\mathcal{R}) \to CX_{n-1}(X,\mathcal{R})/\mathrm{im}\partial\colon CX_{n+1}(X,\mathcal{R}) \to CX_n(X,\mathcal{R}). \end{aligned}$$

Using standard techniques of homological algebra one can prove a coarse map $f\colon X \to Y$ induces a chain map $f_\#\colon CX_n(X,\mathcal{R}) \to CX_n(Y,\mathcal{R})$.

Proposition 7.1.4. *Any coarse map $f\colon X \to Y$ induces a map on homology $f_*\colon HX_n(X,\mathcal{R}) \to HX_n(Y,\mathcal{R})$ for every $n \geq 0$, with the following properties:*

(1) *if $g\colon Y \to Z$, then $(fg)_* = f_*g_*$,*
(2) *if $Y = X$ and $f = \mathrm{Id}$, then f_* is the identity on $HX_n(X,\mathcal{R})$,*
(3) *if $g\colon X \to Y$ and f and g are close, then $f_* = g_*$.*

Applying the last property to a coarse equivalence and its coarse inverse one can easily deduce large scale invariance of coarse homology.

Theorem 7.1.5. *Let X and Y be uniformly discrete, bounded geometry metric spaces. If X and Y are coarsely equivalent, then $HX_n(X,\mathcal{R}) \simeq HX_n(Y,\mathcal{R})$.*

7.2 Uniformly finite homology

In general, computing coarse homology groups is not an easy task. One of the reasons is the fact that the chain groups and the homology groups are both "large", infinite-dimensional linear spaces and *just* linear spaces; that is, there is no natural way to put a norm on these spaces and analytic tools are not available. In certain situations one can overcome this by considering smaller chain groups, which can be equipped with additional analytic structure. This will allow us to use analytic methods to compute homology. This idea was used by Block and Weinberger [34], who introduced uniformly finite homology (see also [35]). We will consider only the case $\mathcal{R} = \mathbb{R}$ and $\mathcal{R} = \mathbb{Z}$.

Definition 7.2.1. Let X be a uniformly discrete, bounded geometry metric space. The *uniformly finite chain groups* $C_n^{\mathrm{uf}}(X, \mathcal{R})$ consist of those coarse chains $c \in CX_n(X, \mathcal{R})$, $c = \sum c_{\bar{x}} \bar{x}$, which satisfy

$$\sup_{\bar{x} \in X^{n+1}} |c_{\bar{x}}| < \infty.$$

Clearly, the definition can be extended to any abelian group with a norm. We consider the same differential as before and we leave it to the reader to check that a differential of a uniformly finite chain is again a uniformly finite chain.

Definition 7.2.2. Let X be a uniformly discrete bounded geometry metric space. The *uniformly finite homology groups* are defined by setting

$$\begin{aligned} &H_n^{\mathrm{uf}}(X, \mathcal{R}) \\ &\quad = \ker \partial\colon C_n^{\mathrm{uf}}(X, \mathcal{R}) \to C_{n-1}^{\mathrm{uf}}(X, \mathcal{R}) / \mathrm{im}\, \partial\colon C_{n+1}^{\mathrm{uf}}(X, \mathcal{R}) \to C_n^{\mathrm{uf}}(X, \mathcal{R}). \end{aligned}$$

The chain groups and the homology groups themselves are still "large" linear spaces. However, the condition in Definition 7.2.1 allows us to put a natural norm on $C_n^{\mathrm{uf}}(X, \mathbb{R})$ as follows:

$$\|c\| = \sup_{\bar{x} \in X^{n+1}} |c_{\bar{x}}|.$$

We can thus identify the uniformly finite 0-chains with the space $\ell_\infty(X)$ of bounded functions on X. We will be interested in a particular class in $H_0^{\mathrm{uf}}(X, \mathbb{R})$.

Definition 7.2.3. Let X be a uniformly discrete, bounded geometry metric space. The constant function 1 on X represents a homology class in $H_0^{\mathrm{uf}}(X, \mathbb{R})$ (and in $H_0^{\mathrm{uf}}(X, \mathbb{Z})$), called the *fundamental class* of X and denoted by $[X]$.

One of the main features of uniformly finite homology is the fact that one can relate it to amenability.

Theorem 7.2.4. *Let G be a finitely generated group. The following conditions are equivalent:*

(1) $H_0^{\mathrm{uf}}(G, \mathbb{R}) = 0$,
(2) $[G] = 0$ *in* $H_0^{\mathrm{uf}}(G, \mathbb{R})$,
(3) *G is not amenable.*

We will prove the above theorem only for groups, however it is also true in a broader setting of bounded geometry metric spaces; see [34].

We begin with a lemma, which states that on a group it is sufficient to consider 1-chains of propagation 1.

Lemma 7.2.5. *Let G be a finitely generated group. $H_0^{\mathrm{uf}}(G,\mathbb{R}) = 0$ if and only if for every 0-chain c there exists a 1-chain ψ with propagation $P_\psi = 1$ such that $\partial\psi = c$.*

Proof. The "if" direction is obvious. For the "only if" direction we will use the fact that the word length metric on G agrees with the restriction of the path metric on the Cayley graph. Let $c \in C_0^{\mathrm{uf}}(G,\mathbb{R})$ and let $\phi \in C_1^{\mathrm{uf}}(G,\mathbb{R})$ be such that $\partial\phi = c$. Let $(x,y) \in G \times G$ be a pair of points such that $d(x,y) = r > 1$ and $\phi(x,y) = c_{(x,y)} \neq 0$. Let γ be a geodesic in the Cayley graph of G, realizing the distance between x and y and let x_i, $i = 0,\dots,r$, be the $r+1$ points on this geodesic, $x_0 = x$, $x_r = y$. Define $\psi \in C_1^{\mathrm{uf}}(G,\mathbb{R})$ by putting $\psi(x,y) = 0$ and $\psi(x_i,x_{i+1}) = c(x,y)$ along the above geodesic. Note that

$$\partial\Big(\sum_{i=0}^{r}\phi(x,y)[x_i,x_{i+1}]\Big) = \partial(\phi(x,y)[x,y] = \phi(x,y)([y]-[x]).$$

Performing this operation for every pair (x,y) with $d(x,y) \le P_\phi$ and summing the coefficients obtained for each edge, we obtain a new chain, $\psi \in C_0^{\mathrm{uf}}(G,\mathbb{R})$, which has propagation 1 and satisfies $\partial\psi = c$, as required. The fact that the new chain is uniformly bounded is left as an exercise. □

By virtue of the above lemma, from now on we will understand by $C_1^{\mathrm{uf}}(G,\mathbb{R})$ the space of 1-chains of propagation 1. We need to introduce additional tools to prove Theorem 7.2.4. Similarly as in the Hulanicki–Reiter condition, we note that non-amenability of G is equivalent to the inequality

$$\sum_{x\in G}|f(x)| \le C\sum_{x\in G}\sum_{y\in B(x,1)}|f(x)-f(y)| \tag{7.1}$$

being satisfied for every finitely supported function $f : G \to \mathbb{R}$ and some uniform constant $C > 0$. Indeed, we have

$$\begin{aligned}\sum_{x\in G}\sum_{y\in B(x,1)}|f(x)-f(y)| &= \sum_{g\in G}\sum_{s\in S}|f(g)-f(gs)| \\ &= \sum_{s\in S}\sum_{g\in G}|f(g)-f(gs)| \\ &= \sum_{s\in S}\|f - s\cdot f\|_1.\end{aligned}$$

Let Σ be a finite generating set of G and consider the discrete gradient on the (oriented) Cayley graph $\mathrm{Cay}(G,\Sigma) = (G, E)$, defined by the formula $\nabla f(x,y) = f(y) - f(x)$ when $(x,y) \in E$ and 0 otherwise. One can check easily that it extends to a continuous linear operator $\nabla\colon \ell_1(G) \to \ell_1(E)$, denoted again by ∇. Then inequality (7.1) states that

$$\frac{1}{C}\|f\|_1 \leq \|\nabla f\|_1,$$

i.e., ∇ has closed range.

Lemma 7.2.6. *The following conditions are equivalent:*

(1) *inequality* (7.1) *holds for all finitely supported* $f : G \to \mathbb{R}$,

(2) *inequality* (7.1) *holds for all non-negative finitely supported* $f : G \to \mathbb{R}$.

The nontrivial direction follows simply from the triangle inequality. Consider now the map $\bar{\partial}\colon \ell_\infty(E) \to \ell_\infty(G)$, defined by

$$\bar{\partial}\psi(x) = \sum_{y \in B(x,1)} \psi(y,x) - \psi(x,y)$$

for any $\psi \in \ell_\infty(E)$. Then $\bar{\partial}$ is a continuous, linear operator. Note that the difference between elements of $\ell_\infty(E)$ and $C_1^{\mathrm{uf}}(G)$ is the presence of orientation: if $\psi \in C_1^{\mathrm{uf}}(E)$, then

$$\psi(x,y) = -\psi(y,x)$$

and

$$\bar{\partial}\psi(x) = 2\partial\psi(x).$$

The following duality is a fundamental tool.

Lemma 7.2.7. *The formula*

$$\langle \bar{\partial}\psi, f\rangle = \langle \psi, \nabla f\rangle$$

holds for any $\psi \in C_1^{\mathrm{uf}}(G)$ *of propagation* 1 *and* $f \in \ell_1(G)$.

Proof. By density, it is enough to prove the equality for f finitely supported. We have

$$\langle \bar{\partial}\psi, f\rangle = \sum_{x\in G} \bar{\partial}\psi(x) f(x) = \sum_{x\in G} \Big(f(x) \sum_{y\in G} \psi(y,x) - \psi(x,y)\Big).$$

If we change the roles of x and y we obtain

$$\begin{aligned}\langle \bar{\partial}\psi, f\rangle &= \sum_{y\in G}\Big(f(y)\sum_{x\in B(y,1)}\psi(x,y)-\psi(y,x)\Big)\\ &= \sum_{y\in G}\Big(-f(y)\sum_{x\in G}\psi(y,x)-\psi(x,y)\Big).\end{aligned}$$

Thus

$$\begin{aligned}2\langle \bar{\partial}\psi, f\rangle &= \sum_{x,y\in G}(f(x)-f(y))(\psi(y,x)-\psi(x,y))\\ &= \sum_{x,y\in G}(f(x)-f(y))2\psi(y,x)\\ &= 2\langle \psi, \nabla f\rangle\end{aligned}$$

since $\psi(y,x) = -\psi(x,y)$. $\square$

Proof of Theorem 7.2.4. (1) $\Longrightarrow$ (2) is obvious. To prove (2) $\Longrightarrow$ (3), let $\psi \in C_1^{\mathrm{uf}}(G)$ be such that $\partial\psi = 1_G$. For a finitely supported, positive function $f : G \to \mathbb{R}$ we have

$$\|f\|_1 = \langle f, 1_G\rangle = \langle f, \partial\psi\rangle = \frac{1}{2}\langle \nabla f, \psi\rangle \leq \frac{1}{2}\|\nabla f\|_1\ \|\psi\|_\infty.$$

The number $\|\psi\|_\infty$ does not depend on f and thus G is not amenable by Lemma 7.2.6.

(3) $\Longrightarrow$ (1). If G is not amenable then it satisfies the inequality (7.1), which implies that the image of ∇ is closed and that ∇ is an isomorphism between $\ell_1(G)$ and the image of ∇, $\mathrm{Im}(\nabla) \subseteq \ell_1(E)$. Thus there is a continuous inverse map $\nabla^{-1}\colon \mathrm{Im}(\nabla) \to \ell_1(G)$. There exists an adjoint map $(\nabla^{-1})^*\colon \ell_\infty(G) \to \mathrm{Im}(\nabla)^*$, where the space $\mathrm{Im}(\nabla)^*$ is the continuous dual of $\mathrm{Im}(\nabla)$, as in the following diagrams:

$$\begin{tikzcd} \ell_\infty(E) \arrow[d, "i^*"'] & \\ \mathrm{Im}(\nabla)^* & C_0^{\mathrm{uf}}(G). \arrow[l, "(\nabla^{-1})^*"'] \end{tikzcd}$$

Here the top diagram is dual to the bottom one.

$$\begin{tikzcd} \mathrm{Im}(\nabla) \arrow[r, shift left, "\nabla^{-1}"] \arrow[d, hook, "i"'] & \ell_1(G) \arrow[l, shift left, "\nabla"] \\ \ell_1(E) & \end{tikzcd}$$

Now, given any uniformly finite 0-chain η, we will construct a uniformly finite 1-chain which bounds η. Since $\eta \in \ell_\infty(G)$, we consider the element $(\nabla^{-1})^*\eta \in \mathrm{Im}(\nabla)^*$. $(\nabla^{-1})^*\eta$ is then a continuous functional on a closed subspace of $\ell_1(E)$ and, using the Hahn–Banach theorem, we can extend it to a continuous functional ϕ_η on $\ell_1(G)$, so that ϕ_η is an element of $\ell_\infty(E)$. We are almost done, except that ϕ_η might not be anti-symmetric. We fix this by setting

$$\psi_\eta = \frac{\phi_\eta - \phi_\eta^T}{2},$$

where $\phi^T(x, y) = \phi(y, x)$. Then ψ_η is a uniformly finite 1-chain, and the proof will be finished once we show that $\partial\psi_\eta = \eta$. Indeed, we have

$$\begin{aligned}
\partial\psi_\eta(x) &= \langle \partial\psi_\eta, 1_x\rangle \\
&= \langle \psi_\eta, \nabla 1_x\rangle \\
&= \frac{1}{2}(\langle \phi_\eta, \nabla 1_x\rangle - \langle \phi_\eta^T, \nabla 1_x\rangle) \\
&= \langle \phi_\eta, \nabla 1_x\rangle \\
&= \langle (\nabla^{-1})^*\eta, \nabla 1_x\rangle \\
&= \langle \eta, \nabla^{-1}\nabla 1_x\rangle \\
&= \langle \eta, 1_x\rangle \\
&= \eta(x).
\end{aligned}$$

Thus for any 0-chain η we have constructed a uniformly bounded 1-chain ψ_η, such that $\partial\psi_\eta = \eta$, which means that $H_0^{\mathrm{uf}}(G, \mathbb{R}) = 0$. □

7.3 Eilenberg swindles and Ponzi schemes

There is a convenient description of the vanishing of the fundamental class in terms of Eilenberg swindles.

Let us consider the uniformly finite homology groups of G with integral coefficients and examine the question of how to ensure that the fundamental class $[G]$ vanishes in $H_0^{\mathrm{uf}}(G, \mathbb{Z})$; that is, $1_G = \partial\psi$ for some uniformly finite 1-chain ψ. The class $[G]$ is just the sum of all 0-simplices, each of which appears with coefficient 1. First let us try to "kill" a single 0-simplex $[x]$.

Consider a sequence of pairwise distinct points $\{x_i\}_{i=\in\mathbb{N}}$ with $d(x_i, x_{i+1}) = 1$,

$d(x_i, e) \to \infty$ and $x_0 = x$. We observe that

$$\begin{aligned}[x] &= [x_0] - [x_1] + [x_1] - [x_2] + [x_2] + \cdots \\ &= \partial[x_1, x_0] + \partial[x_2, x_1] + \cdots \\ &= \partial\Big(\sum_{i=1}^{\infty}[x_{i+1}, x_i]\Big).\end{aligned}$$

The simplex $[x]$ is therefore a boundary of an infinite "tail". We want to kill the fundamental class $[G]$ by attaching a tail to each point.

Assume now we have such tails for two different points $x, y \in G$. If, at some point, the two tails pass through the same edge, then this edge contributes twice to a chain which kills the fundamental class. This means that if we can assign to each $x \in G$ a tail t_x in such a way that, for each edge, the total number of tails passing through it is uniformly bounded, then

$$\sum_{x \in X} t_x \in C_1^{\mathrm{uf}}(G, \mathbb{Z}) \quad \text{and} \quad \partial\Big(\sum_{x \in X} t_x\Big) = [G].$$

It follows that $[G]$ vanishes in $H_0^{\mathrm{uf}}(G, \mathbb{Z})$. The process of attaching the tails to x can be viewed as pushing the coefficient 1 at $[x]$ off to infinity and is known as an *Eilenberg swindle*. It turns out that this is not only a convenient way to kill the fundamental class, it is the only one. The next statement says that if the fundamental

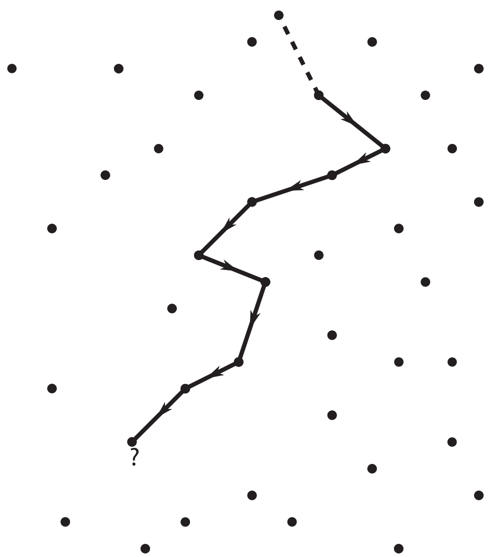

Figure 7.1. A single tail attached to the point $x \in X$.

class vanishes in $H_0^{\mathrm{uf}}(G,\mathbb{Z})$, then one can always reconstruct tails with the above properties.

Proposition 7.3.1. *Let X be a uniformly discrete, bounded geometry metric space and let $c=\sum_{x\in X} c_x[x]$, where $c_x\in\mathbb{Z}$, $c_x\geq 0$ for all $x\in X$, be such that $[c]=0$ in $H_0^{\mathrm{uf}}(X,\mathbb{Z})$. Then, for every $x\in X$ with $c_x>0$, there exists $t_x\in C_1^{\mathrm{uf}}(X,\mathbb{Z})$ satisfying $\partial t_x=x$ and $\sum_{x\in X} t_x\in C_1^{\mathrm{uf}}(X,\mathbb{Z})$.*

Proof. Let $\psi\in C_1^{\mathrm{uf}}(X,\mathbb{Z})$ satisfy $\partial\psi=c$. We start with a single point $x\in X$ such that $c_x=\sup_{y\in X} c_y$. Since $\sum_{y:d(y,x)=1}\psi(y,x)=c_x$, we conclude that there is $x_1\in X$ such that $d(x,x_1)=1$ and $\psi(x_1,x)>0$. The point x_1 gives the first point on our tail t_x and we proceed similarly. Since $\sum_{y:d(y,x_1)=1}\psi(y,x_1)>0$, we can conclude again that there is a point $x_2\in X$ such that $d(x_1,x_2)=1$ and $\psi(x_2,x_1)>0$. We continue in this fashion to obtain an infinite sequence of points $\{x_i\}_{i=1}^{\infty}$. The tail t_x is then defined by

$$t_x=[x_1,x]+\sum_{i=1}^{\infty}[x_{i+1},x_i],$$

and it follows that $\partial t_x=x$.

Now, to construct a tail for a different point, consider the 1-chain $\psi-t_x$. Then $\partial(\psi-t_x)$ satisfies the hypothesis of the proposition, and we can apply the above argument to construct a tail t_y for the point $y\in X$, for which the coefficient $\partial(\psi-t_x)(y)$ is the largest.

Continuing in this fashion we construct a tail t_x for every $x\in X$ for which $c_x>0$. Observe that $\sum_{x\in X} t_x$ is a uniformly finite 1-chain, since every time we constructed a tail, we subtracted it from ψ before constructing the next one. Therefore the simplices that appear in $\sum t_x$ are a subset of those appearing in ψ. □

Example 7.3.2. Let T be an infinite tree with each vertex of degree 3. Then the chain in Figure 7.2 bounds the fundamental class.

Example 7.3.3. Consider $\mathbb{Z}$ with the natural metric. By Theorem 7.2.4, $[\mathbb{Z}]\neq 0$ in uniformly finite homology. To interpret this fact geometrically we look closely at the process of attaching tails to points of $\mathbb{Z}$. In $\mathbb{Z}$ there are only two directions to infinity. At 0 we attach a tail t_0 which escapes to $-\infty$. If we now try to attach a tail to 1 and allow it to escape to the left, the coefficients of edges on the right of 1 increase by 1. The same happens every time we attach a tail to a positive integer n. It is easy to see that such a chain cannot be uniformly bounded. It is also clear that we have to let infinitely many tails escape in each direction to ensure that the coefficients of edges are finite.

It also turns out that there is no difference in vanishing of the fundamental class or the whole 0-dimensional uniformly finite homology, with either real or integral coefficients.

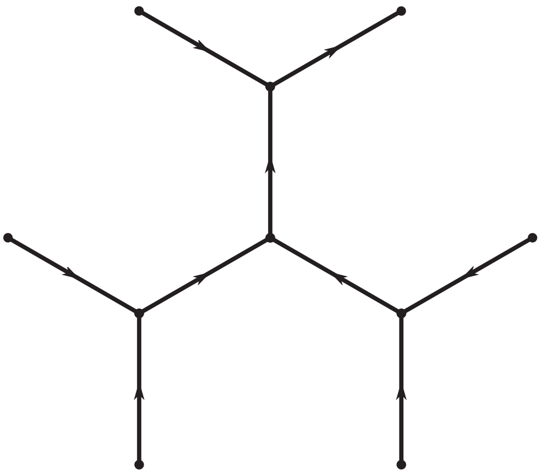

Figure 7.2. A uniformly finite 1-chain which kills the fundamental class on a 3-valent tree. At every vertex there are two edges going in and one going out. The boundary thus is 1 at every vertex.

Proposition 7.3.4. *Let G be a finitely generated group. The following conditions are equivalent:*

(1) $[G] = 0$ *in* $H_0^{\mathrm{uf}}(G, \mathbb{R})$,
(2) $[G] = 0$ *in* $H_0^{\mathrm{uf}}(G, \mathbb{Z})$,
(3) $H_0^{\mathrm{uf}}(G, \mathbb{Z}) = 0$.

Proof. The implications (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are obvious.

To prove (1) $\Longrightarrow$ (2) let S be a finite generating set for G and consider a uniformly finite 1-chain ψ with real coefficients such that $\partial\psi = [G]$. Then $\partial((\#S + 1)\psi) = (\#S + 1)[G]$, and we construct a new chain

$$\phi(x, y) = (\#S + 1)\lfloor|\psi(x, y)|\rfloor \ \mathrm{sign}(\psi(x, y)).$$

Clearly, $\phi \in C_1^{\mathrm{uf}}(G, \mathbb{Z})$. Note also that there is at most $\#S$ coefficients of ψ which contribute to each coefficient of $\partial\phi$. This observation yields

$$\partial\phi(x) = \sum_{y, d(y,x)=1} \phi(y, x) \geq (\#S+1)\Big(\sum_{y, d(y,x)=1} \psi(y, x)\Big) - \#S \geq \#S + 1 - \#S \geq 1.$$

Then application of Proposition 7.3.1 gives the vanishing of $[G]$ in $H_0^{\mathrm{uf}}(G, \mathbb{Z})$.

To show (2) $\Longrightarrow$ (3), consider tails $\{t_x\}_{x\in X}$ constructed as in the proof of Proposition 7.3.1 for the fundamental class. Let $c = \sum_{x\in X} c_x[x]$ be a uniformly finite chain. The 1-chain ψ_c given by

$$\psi_c = \sum_{x \in X} c_x t_x$$

satisfies $\partial \psi_c = c$ and is a uniformly finite 1-chain. Indeed, for $y_1, y_2 \in X$ satisfying $d(y_1, y_2) = 1$, let $A(y_1, y_2)$ be the set of those $x \in X$ for which t_x passes through the edge (y_1, y_2). Let

$$N = \sup_{d(y_1,y_2)=1} \#A(y_1, y_2) < \infty,$$

and observe that

$$|\psi_c(y_1, y_2)| = \Big| \sum_{x \in A(y_1,y_2)} c_x t_x \Big| \le \sum_{x \in A(y_1,y_2)} |c_x| \le \#A(y_1, y_2)\|c\|_{\ell_\infty} \le N\, \|c\|_{\ell_\infty}.$$

□

The proof of the last implication gives another interpretation of the tails. These tails form a sort of a basis, which allows one to kill any other uniformly finite 0-chain by taking linear combinations. This also explains the fact that vanishing of the fundamental class suffices to conclude that the whole 0-dimensional homology group is trivial.

In Chapter 1 we briefly discussed the question of whether given a quasi-isometry $f : X \to Y$ of discrete metric spaces, there exists a bi-Lipschitz equivalence $g : X \to Y$ such that f and g are close. This question turns out to have a complete answer in terms of uniformly finite homology.

Theorem 7.3.5 ([237]). *Let X and Y be uniformly discrete metric spaces with bounded geometry and $f : X \to Y$ be a quasi-isometry. There is a bi-Lipschitz map $g : X \to Y$ close to f if and only if $f_*[X] = [Y]$.*

The above facts can be immediately used to show the following property of finite index subgroups of amenable groups, observed in [82]. The following proof is due to S. Weinberger.

Corollary 7.3.6. *Let G be an amenable subgroup and let H be a subgroup of finite index. The inclusion $i : H \to G$ is not close to a bi-Lipschitz equivalence.*

Proof. Assume the contrary. According to Theorem 7.3.5,

$$[H] = i_*[H] = [G].$$

On the other hand, if we denote by $(G : H)$ the index of H in G, then

$$[G] = (G : H)[H].$$

Combining the two equalities we obtain $(G : H) = 1$. □

Ponzi schemes. There is another interpretation of the Eilenberg swindle. Consider the following business model. A person A finds two investors, B_1 and B_2, who invest a certain amount of money with A. Then both B_1 and B_2 find two new investors each and the process is repeated, i.e., B_1 finds C_1 and C_2, while B_2 finds C_3 and C_4. Clearly, all the C_i's lose money until they find investors of their own and convince them to invest with them. Such a scheme is a version of the well-known Ponzi scheme. The idea in such schemes is that new investors pay for the old investor's debt. However, the number of new investors has to grow very rapidly to make up for the increasing losses. In the real world, where the set of potential investors is finite, the scheme is bound to collapse. However, the reader might have already noticed that the construction of the above scheme, viewed as a directed graph where arrows depict the flow of money, is identical to the construction of tails and Eilenberg swindles on the 3-valent, infinite tree. This is a more general phenomenon. For every non-amenable space of investors there is a Ponzi scheme which is a perfectly legitimate business model: everyone makes a profit.

7.4 Aperiodic tiles and non-amenable spaces

In this section we will describe an application of uniformly finite homology and construct aperiodic tiles for certain open Riemannian manifolds. We will be mainly interested in the case of an open manifold M which is a regular covering space of a compact Riemannian manifold N. In that case we will assume that M is equipped with a metric and a triangulation, both lifted from the metric and the triangulation on N and both invariant under the action of the group of deck transformations.

Observe that the definition of amenability (Definition 3.1.1) applies to any metric space, not only to groups with word length metrics. We will call M amenable if some (equivalently, every) discrete net of bounded geometry in M is amenable in the sense of Definition 3.1.1. In particular a regular cover M of a compact manifold N is amenable if and only if the group of deck transformations is amenable.

Let M be a covering space as above.

(1) A *tile* T of M is a compact triangulated n-submanifold of M with boundary homeomorphic to some closed ball in M;

(2) a *tiling* of M is a decomposition $M = \bigcup_i T_i$, where each T_i is a tile and different tiles can intersect only along their boundary;

(3) a *prototile* is a compact triangulated n-manifold T with boundary. A set $\mathcal{T}$ is a set of prototiles of a given tiling if it has the property that each tile in the tiling is isometric to one of the prototiles, and this isometry preserves the triangulation.

We will be interested in finite sets of prototiles which only admit certain irregular tilings, and we need to put certain restrictions on how different prototiles can fit together. The boundary of a prototile T_i is a triangulated $(n-1)$ manifold and contains a collection of faces F_{ij}, along which two tiles can intersect.

Definition 7.4.1. Let $\mathcal{T} = \{T_i\}$ be a finite collection of prototiles (i.e., a protoset) and for each i let $\{F_{ij}\}_j$ be the collection of faces of T_i. Let $\mathcal{F} = \{F_{ij}\}_{i,j}$. An *opposition function* is a function

$$\mathcal{O}\colon \mathcal{F} \to \mathcal{F}$$

such that $\mathcal{O}(F_{ij}) = F_{i'j'}$ only if F_{ij} and $F_{i'j'}$ are isometric.

The opposition function is simply a matching rule for the tiles. Such matching rules can be implemented geometrically by modifying the two matching sides so that only these two can fit together, similarly to pieces of a jigsaw puzzle. For instance, if $\mathcal{O}(F_{ij}) = F_{i'j'}$, then we can make an indentation in T_i on the side F_{ij} and put a matching peg in $T_{i'}$ on the side $F_{i'j'}$. The matching faces have to be isometric, but not all isometric faces have to match.

It is not hard to come up with examples of tiles and tilings, in fact many examples can be observed in our everyday lives.

Example 7.4.2. One can tile $\mathbb{R}^2$ by simplices by first tiling $\mathbb{R}^2$ by copies of a square in the obvious way and dividing each square into two 2-simplices by a diagonal.

The reader can easily come up with other examples of tilings for different spaces. It is likely that most of them will have the property of being symmetric in one way or another, or have a finite pattern repeated throughout the tiling. The question we want to address now is how to construct sets of prototiles without such properties.

Definition 7.4.3. A tiling of M by tiles from a finite protoset $\mathcal{T}$ is *periodic* if there exists a cocompact isometric action of some group G of M that preserves the tiling. The tiling is said to be *aperiodic* if it is not periodic.

A protoset $\mathcal{T}$ for M is *aperiodic* if any tiling of M by elements of $\mathcal{T}$ is aperiodic.

Aperiodic sets of prototiles are not easy to construct. The first finite set of aperiodic tiles of the Euclidean plane was constructed by Berger [32] in 1966 and consisted of over $2\cdot 10^4$ tiles. Since then various other constructions have emerged, in particular one of the most famous sets of tiles, Penrose's darts and kites [202]. We refer to [112], [204] for an overview of different perspectives and techniques.

In settings other than Euclidean, less is known. An interesting case is that of a hyperbolic plane, for which a single aperiodic tile was constructed by Penrose, see Figure 7.4. The proof of aperiodicity of this tile is homological in nature: it turns out that the underlying reason is the existence of Eilenberg swindles. This construction was generalized significantly in [34].

In order to use uniformly finite homology to construct aperiodic tiles we need to define a notion of an unbalanced tile.

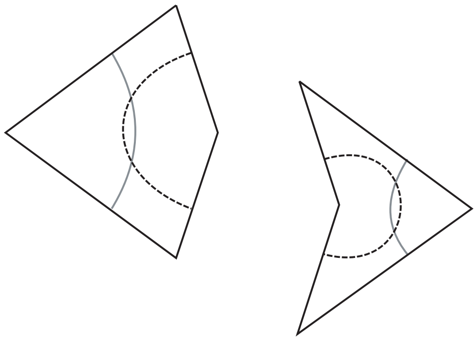

Figure 7.3. Penrose's aperiodic tiles in the Euclidean plane, known as "darts and kites", with their matching rules. The patterns on the tiles are required to match.

Definition 7.4.4. A finite set of tiles $\mathcal{T}$ is said to be *unbalanced* if there exists a weight function $\omega\colon \mathcal{F} \to \mathbb{Z}$ such that

(1) $\omega(\mathcal{O}(F)) = -\omega(F)$ for every face $F \in \mathcal{F}$,
(2) for every prototile $T \in \mathcal{T}$ we have $\sum_{F_j \in \mathcal{F}, F_j \subseteq T} \omega(F_j) > 0$.

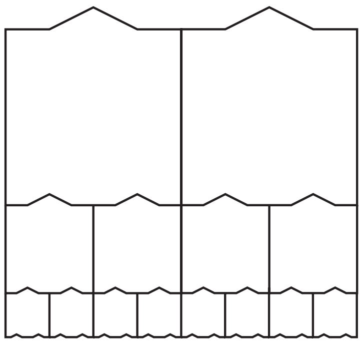

Figure 7.4. Penrose's unbalanced tile in the hyperbolic plane and an associated aperiodic tiling.

From now on we will assume that each prototile has a distinguished point in its interior. The images of these points give a discrete net in M, quasi-isometric to M.

Theorem 7.4.5. *Let M be an amenable manifold. There are no tilings of M by unbalanced tiles.*

Proof. Consider the metric space X whose elements are faces of tiles appearing in the tiling, with the natural distance

$$d_X(F, F') = d_M(x_F, x_{F'}).$$

Here x_F denotes a distinguished point in F, chosen in the interior of F in such a way that for matching faces F, F', belonging to different tiles, the distance satisfies $C \le d_X(F, F') \le K$, for some uniform constants $C, K > 0$. We leave it as an exercise to check that if M is amenable, then so is X. We construct a functional φ, similarly as in the proof of Theorem 3.3.2. Given a sequence of sets $A_n \subseteq X$ such that $\dfrac{\#\partial A_n}{\#A_n} \to 0$, we define

$$\varphi(f) = \lim_\beta \frac{1}{\#A_\beta} \sum_{F \in A_\beta} f(F),$$

where the limit is taken over a weak*-convergent subnet, as in the proof of Theorem 3.3.2. Moreover, we can choose the sets A_n so that for each n there is a collection $T^n = T_i^n$ of copies of prototiles, so that $F \in A_n$ if and only if F is a face of one of the T_i's. We apply this functional to the function obtained by weights on the tiling, where we view $f(F) = \omega(F)$ as an element of the linear space of bounded functions on the set of all faces of the tiling. We have

$$\begin{aligned}
\varphi(f) &= \lim_\beta \frac{1}{\#A_\beta} \sum_{F \in A_\beta} \omega(F) \\
&= \lim_\beta \frac{1}{\#A_\beta} \sum_{T_i^\beta \in T^\beta} \sum_{F \subseteq T_i^\beta} \omega(F) \\
&\ge \lim_\beta \frac{1}{\#A_\beta} \sum_{T_i^\beta} 1 \ge \frac{\#T^\beta}{\#A_\beta}.
\end{aligned}$$

Clearly,

$$\inf_{n \in \mathbb{N}} \frac{\#T^n}{\#A_n} > 0. \tag{7.2}$$

On the other hand,

$$\varphi(f) = \lim_\beta \frac{1}{\#A_\beta} \sum_{T_i^\beta \in T^\beta} \sum_{F \subseteq T_i^\beta} \omega(F).$$

If two tiles in A_n intersect along a face F, then the weights of these faces cancel each other out, and the only faces which do not have a corresponding opposite in A are the ones on the boundary of A. Thus we have

$$\begin{aligned}
\varphi(f) &= \lim_\beta \frac{1}{\#A_\beta} \Big(\sum_{F \subseteq A_\beta \setminus \partial A_\beta} \omega(F) + \omega(\mathcal{O}(F)) \Big) + \sum_{F \in \partial A_\beta} \omega(F) \\
&= \lim_\beta \frac{1}{\#A_\beta} \sum_{F \in \partial A_\beta} \omega(F) \le \lim_\beta C \frac{\#\partial A_\beta}{\#A_\beta},
\end{aligned}$$

where C is such that $|\omega(F)| \leq C$ for any $F \in \mathcal{F}$. By the properties of the sequence A_n, the right-hand side of this inequality converges to 0 and we obtain $\varphi(f) = 0$, contradicting (7.2). □

Since compact spaces are amenable, the above theorem yields

Corollary 7.4.6. *An unbalanced set of prototiles is aperiodic.*

Proof. Let M be tiled by prototiles from $\mathcal{T}$ and assume that there exists a group of isometries which preserves the tiling and such that the quotient is compact. Then the quotient space is also tiled by prototiles from $\mathcal{T}$. In particular, the tiling of the quotient space is unbalanced, and since a compact space is amenable, we obtain a contradiction to Theorem 7.4.5. □

In fact a stronger form of aperiodicity can be easily deduced from the proof: there is no isometric group action on M that preserves the tiling and such that the quotient is amenable. This feature is a consequence of the fact that, for a finite set of tiles, being unbalanced is stronger than being aperiodic. To complete our discussion we will prove the main theorem of this section.

Theorem 7.4.7 ([34]). *Let M be a non-amenable regular covering space of a compact Riemannian manifold of dimension at least* 2. *Then there is an unbalanced set of tiles of M.*

We will first assume that we have a convenient periodic tiling of M and then modify it to obtain aperiodicity. We will deal with the problem of existence of such a periodic tiling in a separate lemma.

Suppose that we have a finite set of tiles $\mathcal{T}$ that tiles M, and let X be the net of distinguished points in tiles mentioned before. Up to quasi-isometry the distance on X is a restriction of a distance on a graph in which we consider two points connected if the tiles share a face. We will additionally assume that the vertices of X are arranged in such a way that each edge pierces exactly one side of a tile that contains that vertex, and that each side of a tile is pierced by an edge.

Lemma 7.4.8. *Let M be a regular covering space of a compact Riemannian manifold. Then there is a finite set of tiles, with the above properties, that tile M.*

Assuming the above lemma we can prove Theorem 7.4.7.

Proof of Theorem 7.4.7. Since M, and consequently X, is non-amenable, there is a uniformly finite chain $\psi \in C_1^{\mathrm{uf}}(X, \mathbb{Z})$ such that $\partial\psi = 1_X$. For each tile T_i in the tiling we construct the weight on the faces by the following rule:

$$\omega(F_{ij}) = \text{the coefficient } \psi(x, y) \text{ of the edge } (x, y) \text{ piercing } F_{ij}, \\ \text{pointing outward from } T_i.$$

Note that since ψ is a chain, ω is well defined. Since ψ is uniformly bounded and has values in $\mathbb{Z}$, there are only finitely many weights that one can attach to the edges. Thus, since there are finitely many prototiles for the tiling, the number of tiles after assigning weights is still finite. Note that since $\psi(x, y) = -\psi(y, x)$, the weight function is antisymmetric with respect to the opposition function, as required. Finally, for every i we have

$$\sum_j \omega(F_{ij}) = \sum_{y:d(y,x_i)=1} \psi(y, x_i) = \partial\psi(x_i) = 1 > 0,$$

where $x_i \in F_i$ is the distinguished point. Therefore, the constructed tiling is unbalanced. $\square$

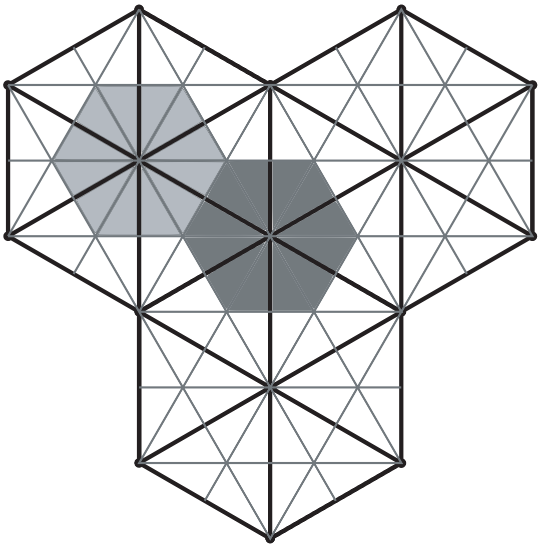

Figure 7.5. The tiling of the Euclidean plane obtained from a barycentric subdivision by the method described in Lemma 7.4.8. The black lines are the edges of the original triangulation τ. The thinner gray edges are the edges of the barycentric subdivision τ'. Two shaded regions are stars of two vertices in τ. They meet along a face which is pierced by an edge from τ.

To complete the proof of Theorem 7.4.7 we need to prove Lemma 7.4.8.

Proof of Lemma 7.4.8. Let M be a regular covering space of a compact Riemannian manifold N and choose a triangulation of the latter. Lifting it to M we obtain a triangulation τ of M. Consider a barycentric subdivision τ' of τ. For each vertex x in τ take the star of x in τ', that is, the union of all simplices in τ' containing v,

$$\mathrm{Star}_{\tau'}(x) = \bigcup_{\sigma_n \in \tau',\, x \in \sigma_n} \sigma_n.$$

By the properties of τ, there exists a finite set K of vertices of τ such that for every vertex y of τ there exists $x \in K$ with $\mathrm{Star}_{\tau'}(y)$ is isometric to $\mathrm{Star}_{\tau'}(x)$. Write $\mathcal{T} = \{\mathrm{Star}(x)\}_{x \in K}$. Now for each edge $e = (x, y)$ in τ let

$$\mathrm{Star}_{\tau'}(e) = \mathrm{Star}_{\tau'}(x) \cap \mathrm{Star}_{\tau'}(y).$$

Then take the prototiles to be the stars of $x \in V$ and faces to be the stars of edges. □

We point out that Theorem 7.4.7 is true in a wider setting than considered here; we refer to [34] for details. We also remark that the argument in Theorem 7.4.7 can be reversed. We leave this as an exercise.

7.5 Coarsening homology theories

This section is independent of the previous material and is directed towards a reader with a slightly stronger background in algebraic topology. We will assume familiarity with simplicial complexes, proper homotopy and generalized homology theories.

We will describe here a general construction, which, given a homology theory h, produces a new theory hx. This new object is well-suited for the study of large scale geometry. The homology theory h will be assumed to be a generalized homology theory on the category of locally compact topological spaces with proper maps (recall that a map is proper if the preimage of every compact set is compact). The construction is due to Roe [207]. Coarse homology discussed in the first section of this chapter is a particular case of this construction.

Definition 7.5.1 (Rips complex). Let X be a locally finite metric space. Fix $R > 0$. The vertices of the *Rips complex* are given by the points of X; a collection of $n+1$ distinct vertices $\{x_0, \dots, x_n\}$ spans an n-simplex if and only if $d(x_i, x_j) \le R$ for all $i, j = 0, \dots, n$.

It is easy to check that the Rips complex is simply the nerve of the cover of X by balls of radius $R/2$. For $R > 0$ we denote the Rips complex by $P_R(X)$. Note that every simplex in $P_R(X)$ is also a simplex in $P_{R'}(X)$ whenever $R' \ge R$. Consequently, for every $n \in \mathbb{N}$ we have a natural simplicial map

$$f_n \colon P_n(X) \to P_{n+1}(X),$$

induced by the inclusion. The sequence of compositions

$$P_1(X) \to \cdots \to P_n(X) \to P_{n+1}(X) \to \cdots$$

induces a sequence of maps in homology and give rise to the direct system

$$h_*(P_1(X)) \to \cdots \to h_*(P_n(X)) \to h_*(P_{n+1}(X)) \to \cdots .$$

For the remainder of this section we will consider only metric spaces in which every uniformly discrete net is locally finite.

Definition 7.5.2. Let X be a metric space and let N be a net in X. The *coarse homology* hx of the space X, associated to the homology theory h, is defined as the direct limit

$$hx_*(X) = \varinjlim h_*(P_n(N)).$$

We will show that the definition of hx does not depend on the choice of the net N, and we establish several natural properties of coarse homology.

Example 7.5.3. Let X be a finite metric space and h the simplicial homology. Then $hx(X) = h(\{\text{point}\})$. Indeed, observe that for sufficiently large n the Rips complex $P_n(N)$ is a single simplex, regardless of the choice of N.

Proposition 7.5.4. *Let X and Y be metric spaces. A coarse map $f: X \to Y$ induces a natural map $f_*: hx_*(X) \to hx_*(Y)$ in coarse homology.*

Proof. Let N be a net in X. Then $f(N)$ is a subset of Y that can be extended to a net in Y. Without loss of generality we can assume that f is onto. Let $\mathcal{B}_X^n$ be the cover of X by balls of radius n, centered at points of N. The collection $f(\mathcal{B}_X^n)$ consisting of images of the elements of $\mathcal{B}_X^n$ under f is uniformly bounded since f is coarse. We can choose a number $k(n)$ such that $f(\mathcal{B}_X^n)$ is inscribed in $\mathcal{B}_Y^{k(n)}$ and such that this inclusion induces a simplicial map

$$f_n: P_n(N) \to P_{k(n)}(f(N)).$$

Similarly construct a map

$$f_{n+1}: P_{n+1}(N) \to P_{k(n+1)}(f(N)),$$

with the additional property that $f(\mathcal{B}_X^{k(n)})$ is inscribed in $\mathcal{B}_Y^{k(n+1)}$. Then the diagram

$$\begin{tikzcd}
P_n(N) \arrow[r] \arrow[d, "f_n"'] & P_{n+1}(N) \arrow[d, "f_{n+1}"] \\
P_{k(n)}(f(N)) \arrow[r] & P_{k(n+1)}(f(N))
\end{tikzcd}$$

commutes, as does the induced diagram in homology. This implies that the limit map $f_* = \varinjlim (f_n)_*$ is a well-defined homomorphism, as required. $\square$

Proposition 7.5.5. *Let $f, g: X \to Y$ be coarse maps which are close. Then $f_* = g_*$.*

Proof. Under the above assumptions, for each $n \in \mathbb{N}$ we can choose $k(n)$ in such a way that f_n and g_n are mapped into the same $\mathcal{B}_Y^{k(n)}$ and are linearly homotopic. Thus for every n, $(f_n)_*$ and $(g_n)_*$ induce the same maps in homology, and this property is preserved by the direct limit. □

Corollary 7.5.6. *Let $f : X \to Y$ be a coarse equivalence. Then $hx_*(X) = hx_*(Y)$.*

Proof. Given f there exists $g : Y \to X$ such that the composition $f \circ g$ is close to Id_X and $g \circ f$ is close to Id_Y. The claim follows from the two previous propositions. □

It follows from the above that, as claimed earlier, the definition of hx does not depend on the choice of the net N. We remark that Corollary 7.5.6 is true for a larger class of maps, *coarse homotopy equivalences*; see [129] for details.

7.6 The coarsening homomorphism

We will now investigate conditions under which the coarse homology groups $hx_*(X)$ are naturally isomorphic to the homology groups $h_*(X)$. We first construct a *coarsening homomorphism*, $c : h_*(X) \to hx_*(X)$.

Let $N \subseteq X$ be a locally finite net and let $\{\varphi_x\}_{x \in N}$ be a partition of unity associated to the cover of X by balls of radius R (see for instance Example 4.3.3), centered at points of N. Define a map $\kappa : X \to P_R(N)$ by the formula

$$\kappa(x) = \sum_{x \in N} \varphi_x B(x, R).$$

This function describes, in barycentric coordinates, a point in the Rips complex $P_R(N)$. In this way, for $R = 1$ we obtain a map $\kappa : h_*(N) \to h_*(P_1(N))$, which, after passing to the direct limit, gives rise to the coarsening homomorphism for h,

$$c : h_*(X) \to hx_*(X).$$

The question we want to address is: *under what conditions is c an isomorphism*? A natural condition in this context is a local topological triviality.

Definition 7.6.1. A metric space X is said to be *uniformly contractible* if for every radius $R > 0$ there exists a number $S = S(R) > 0$ such that for every $x \in X$ the ball $B(x, R)$ is contractible inside $B(x, S)$.

For manifolds, uniform contractibility implies contractibility. Euclidean and hyperbolic spaces are examples of uniformly contractible spaces. More generally, the universal cover of a compact, aspherical manifold is uniformly contractible.

Theorem 7.6.2 ([207]). *Let X be a metric space such that every net in X has bounded geometry. If X is uniformly contractible, then the coarsening homomorphism $c\colon h_*(X) \to hx_*(X)$ is an isomorphism.*

We will first need a lemma on approximation of coarse maps by continuous maps. We equip a simplicial complex with a metric which, restricted to any simplex, is the Euclidean metric on the standard simplex in $\mathbb{R}^n$.

Lemma 7.6.3. *Let X be a finite-dimensional simplicial complex and let Y be a uniformly contractible metric space. Let $f\colon X \to Y$ be a coarse map. Then there exists a continuous map $g\colon X \to Y$ which is close to f. Moreover, if f is already continuous on a subcomplex X', then we may arrange for $f = g$ on X'.*

We leave the proof of the lemma to the reader.

Proof of Theorem 7.6.2. Let $N \subseteq X$ be a net in X and let $\mathcal{U}_1$ be a covering of X by the collection $\{B(x,1)\}_{x\in N}$ of balls of radius 1. By the bounded geometry condition, the Rips complex $P_1(N)$ is a finite-dimensional simplicial complex. The map $\kappa\colon X \to P_1(N)$ is a coarse equivalence and it has a coarse inverse $g_1\colon P_1(N) \to X$. We may also assume that this inverse is continuous and that $g_1 \circ f_1$ is properly homotopic to Id_X. On the other hand, the map $f_1 \circ g_1$ is close to the identity on $P_1(N)$. Similarly, considering a covering $\mathcal{U}_2$ by balls of radius 2 centered at N, we see that there is a map $f_2\colon P_1(N) \to P_2(N)$ and a coarse inverse, $g_2\colon P_2(N) \to X$, of $f_2 \circ f_1$. We again have that $f_2 \circ f_1 \circ g_2$ is properly homotopic to f_2, through a linear homotopy. Continuing in this fashion we obtain a direct system

$$X \xrightarrow{f_1} P_1(N) \xrightarrow{f_2} P_2(N) \xrightarrow{f_1} \cdots,$$

which has the following properties:

(1) every f_i is continuous,

(2) every map $h_i = f_i \circ \ldots \circ f_1$ admits a left proper homotopy inverse g_i,

(3) f_{i+1} is properly homotopic to $h_{i+1} \circ g_i$.

Since the homology theory h_* is invariant under proper homotopy equivalences, it follows that for every $i \in \mathbb{N}$ the induced map $(h_i)_*\colon h_*(X) \to h_*(P_i(N))$ is an isomorphism onto the image of $(f_i)_*$. This shows that the map c_*, as an induced map on the direct limit, is an isomorphism as well. □

We remark that the bounded geometry hypothesis in Theorem 7.6.2 cannot be dropped. Dranishnikov, Ferry and Weinberger [73] constructed a uniformly contractible manifold which does not satisfy this assumption and for which the map c_* is not injective. See also [71].

Remark 7.6.4. For practical purposes it is sometimes convenient to consider a weaker property than the coarsening homomorphism being an isomorphism. Let $\mathcal{M}$ be an open orientable manifold and let $[\mathcal{M}] \in H_n^{lf}(\mathcal{M})$ be the fundamental class in the locally finite homology of M. In [99] a manifold is called *macroscopically large* if the class $[\mathcal{M}]$ survives the coarsening process; that is,

$$c([\mathcal{M}]) \neq 0$$

in $HX_n(\mathcal{M})$. This means that the local topological data gives some information about the large scale geometry as well. For discussion and applications see [99], [123], [185].

One application of coarse homology is a lower bound on the asymptotic dimension of the Euclidean space. Let H denote the locally finite homology.

Proposition 7.6.5. *Let X be a bounded geometry metric space with* asdim $X \leq n$. *Then $HX_k(X) = 0$ for all $k \geq n+1$.*

To prove the above fact we need to improve our definition of coarse homology. Let X be a proper metric space and let h be a homology theory as before. Define an anti-Čech system for X to be a sequence of covers $\mathcal{U} = \{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of X such that

(1) each cover $\mathcal{U}_n$ is uniformly bounded,

(2) the Lebesque number λ_{n+1} of $\mathcal{U}_{n+1}$ satisfies $\lambda_{n+1} \geq R_n$ for some $R_n > 0$,

(3) $\lim_{n\to\infty} R_n = \infty$.

Denote by $|\mathcal{U}_n|$ the nerve of the cover $\mathcal{U}_n$. It is not hard to modify the arguments presented in this section to check that one can define $hx_*(X) = \varinjlim h_*(|\mathcal{U}_n|)$ for any anti-Čech system and obtain coarse homology groups isomorphic to the ones defined earlier, using Rips complexes.

If X satisfies asdim $X \leq n$, then the coverings that give this upper bound can be modified to have appropriate Lebesque numbers, so that they can form an anti-Čech system but at the same time have multiplicity bounded above by $n+1$. This implies that all nerves $|\mathcal{U}_n|$ have simplices only of dimension at most n, which gives Proposition 7.6.5.

To use the above fact to obtain a lower bound on the asymptotic dimension of $\mathbb{R}^n$, we observe that since $\mathbb{R}^n$ is uniformly contractible, its coarse homology and locally finite homology agree in all positive degrees. The locally finite homology of $\mathbb{R}^n$ is isomorphic to the homology of the n-sphere and does not vanish in degree n. By Proposition 7.6.5, asdim $\mathbb{R}^n \geq n$.

Exercises

Exercise 7.1. Show that the chain ψ in the proof of Lemma 7.2.5 is uniformly bounded.

Exercise 7.2. Using tails show that $HX_0(G) = 0$ for any infinite group G.

Exercise 7.3. Characterize the best constant in the isoperimetric inequality using norms of uniformly finite 1-chains.

Exercise 7.4. Prove the converse to Theorem 7.4.7: if M has an unbalanced tiling then M is not amenable.

Exercise 7.5. Show that any finite set of tiles $\mathcal{T}$ for $\mathbb{R}$ admits a periodic tiling.

Notes and remarks

The large scale geometric approach to index theory, as mentioned earlier, was developed by Roe in [207]. We refer the reader to [129], [208], [209] for an overview of the material. Roe's work on index theory allowed us to formulate the coarse Baum–Connes conjecture, which will be discussed in the next chapter. In particular, this development led to the recent remarkable progress on the coarse Baum–Connes conjecture and the Novikov conjecture, surveyed in the next chapter.

An axiomatic approach to coarse homology was given by Mitchener [170], who also proved that such homology theories agree on "coarse CW-complexes", a fact parallel to a well-known theorem in algebraic topology.

Interesting constructions of aperiodic tiles for open manifolds can be found in [159], [174].

A detailed discussion of analytic K-homology, which is a certain generalized homology theory to which the results of the last section apply, can be found in [131]. In particular the reader will find there a discussion of the coarsened K-homology.

A Mayer–Vietoris sequence is proved in [128] for the related coarse cohomology, not discussed here.

Chapter 8
Survey of applications

8.1 Topological rigidity

A fundamental question in topology is the classification of topological spaces. This task is not a reasonable one in such generality and one cannot hope to be able to classify *all* topological spaces. However, one can sometimes restrict to a class of "nicer" spaces with an additional structure and hope that the extra information will allow one to draw stronger conclusions. Here we will focus on the class of Riemannian manifolds. By topological rigidity we mean that if two (Riemannian) manifolds are roughly the same geometrically, then they are exactly the same topologically, i.e., homeomorphic. For classification purposes this kind of rigidity is of great importance.

Let us try to decide what "roughly the same geometrically" should mean. Assume that we are given two complete Riemannian manifolds. A natural notion of a weak equivalence is the one we have been studying here, namely coarse equivalence. It is tempting to test the following speculation: *if M and N are coarsely equivalent, then M and N are homeomorphic*. This statement is clearly false. In order to disprove it, it suffices to take two non-homeomorphic compact manifolds. To make a statement which does not have an immediate counterexample, we need to add an additional condition to trivialize the local topology and restrict our question to the class of uniformly contractible manifolds. The universal cover of a compact, aspherical manifold is a typical example of a uniformly contractible manifold. Recall that a manifold M is said to be *aspherical* if its universal cover is contractible, which is the same as saying that all the homotopy groups $\pi_n(M)$ vanish for $n \geq 2$. Indeed, the universal cover is simply connected by definition, and the equality of higher homotopy groups of M and its universal cover follows from the long exact sequence of a fibration. We now state

Conjecture 8.1.1 (Coarse rigidity conjecture). *Let M and N be uniformly contractible manifolds. If M and N are coarsely equivalent, then M and N are homeomorphic.*

The conjecture is obviously false without the uniform contractibility assumption. Applied to universal covers, the above coarse rigidity conjecture would state that if X_1 and X_2 are compact aspherical manifolds, with $\pi_1(X_1)$ coarsely equivalent to $\pi_1(X_2)$, then X_1 and X_2 are in fact homeomorphic. The coarse rigidity conjecture is a geometric version of the classical Borel conjecture.

Conjecture 8.1.2 (Borel conjecture). *Let X, Y be compact, aspherical manifolds. If X and Y are homotopy equivalent, then X and Y are homeomorphic.*

Assume that the coarse rigidity conjecture holds. Then, given compact aspherical manifolds X and Y which are homotopy equivalent, we note that $\pi_1(X)$ is isomorphic and, in particular coarsely equivalent, to $\pi_1(Y)$. By the Milnor–Švarc lemma (1.3.13) we know that the universal covers $\widetilde{X}$ and $\widetilde{Y}$ of X and Y, respectively, are coarsely equivalent. The coarse rigidity conjecture would then imply that they are homeomorphic.

At present no counterexamples to the Borel conjecture are known. Many important results on the Borel conjecture have been obtained by Farrell and Jones [89], [91], [92], [93] and more recently by Bartels and Lück in [20], where the Borel conjecture was proved for manifolds with CAT(0) and hyperbolic groups. It is known that the Borel conjecture fails if one considers smooth manifolds and diffeomorphisms instead of homeomorphisms; counterexamples were constructed by Farrell and Jones [90].

Let us now consider a weaker statement, namely the stable Borel conjecture. Stabilization in topology usually refers to multiplying a topological space by a copy of the Euclidean space.

Conjecture 8.1.3 (Stable Borel conjecture). *Let X, Y be compact, aspherical manifolds. If X and Y are homotopy equivalent, then there exists $n \in \mathbb{N}$ such that $X \times \mathbb{R}^n$ and $Y \times \mathbb{R}^n$ are homeomorphic.*

Farrell and Hsiang proved the stable Borel conjecture for non-positively curved manifolds [88]. In Chapter 2 we introduced decomposition complexity of a metric space. The following has been proved by Guentner, Tessera and Yu [117].

Theorem 8.1.4. *Let M be a compact, aspherical manifold M such that $\pi_1(M)$ has finite decomposition complexity. Then the stable Borel conjecture is true for M.*

The Novikov conjecture can be viewed as an infinitesimal version of the Borel conjecture. For aspherical manifolds it has a particularly concise statement.

Conjecture 8.1.5 (Novikov conjecture). *The rational Pontryagin classes of a compact aspherical manifold are homotopy invariants.*

Important results on the Novikov conjecture have in particular been obtained by Miščenko [168], Kasparov [142], Connes [59], Connes–Gromov–Moscovici [61], Connes–Moscovici [62], Dranishnikov–Ferry–Weinberger [74], Higson–Kasparov [126] and Yu [249], [250].

8.2 Geometric rigidity

Scalar curvature is a very weak measure of the local geometry of a manifold. It can be defined in several ways, here we will use the following description.

Definition 8.2.1. Let M be a Riemannian manifold of dimension m. Then for any point $x \in M$ the *scalar curvature* $\kappa(x)$ at x is defined by the following formula:

$$\frac{\operatorname{Vol}(B_M(x,r))}{\operatorname{Vol}(B_{\mathbb{R}^m}(r))} = 1 - \frac{\kappa(x)}{6(m+2)} r^2 + \cdots .$$

Here $B_M(x,r)$ denotes the metric ball with center x and radius r.

Scalar curvature carries information about the comparison of volumes of small balls in the m-manifold M and the Euclidean space $\mathbb{R}^m$. In particular, if κ is positive, balls of small radius in M are smaller in volume than balls of the same radius in $\mathbb{R}^m$.

A fundamental problem in geometry is whether a given manifold can carry a metric of positive scalar curvature. Remarkable work on this problem has been done by Gromov and Lawson [111], Schoen and Yau [221], Stolz [225] and Rosenberg [212].

The notion of uniform contractibility is one of many forms of *largeness* of a non-compact Riemannian manifold. Largeness conditions allow one to transfer certain local information to large scale. For instance, as we have seen in the last chapter, uniform contractibility allows one to translate homological data from small to large scale. However, uniform contractibility is a restrictive condition in the realm of open manifolds, and we need conditions which are slightly more flexible.

Definition 8.2.2 ([99]). A non-compact manifold M of dimension m is said to be *macroscopically large* if the coarsening homomorphism $c\colon H_m^{lf}(M) \to HX_m(M)$ preserves the fundamental class of the manifold, i.e.,

$$c([M]) \neq 0 \quad \text{in } HX_m(M).$$

Various other notions of largeness were introduced and studied by Gromov [106]. A more detailed study of macroscopical largeness appeared in [46], [123]. The following conjecture can be viewed as a "geometric rigidity conjecture".

Conjecture 8.2.3 (Geometric rigidity conjecture). *If M is macroscopically large, then M cannot be "microscopically small", i.e., carry a metric with uniformly positive scalar curvature.*

Here uniformly positive means that it is bounded from below by a positive constant. This conjecture was verified in a number of cases, in particular in [99] the following was proved.

Theorem 8.2.4. *Let M be a macroscopically large manifold with subexponential volume growth. Then M cannot be microscopically small.*

Volume growth of a manifold is defined exactly as volume growth for groups (see 3.1.8). The proof of the above theorem proceeds by showing that subexponential growth for a manifold M implies that M admits a coarse embedding into a Hilbert space. This together with a result on the coarse Baum–Connes conjecture, which will be recalled in the next section, gives the claim.

8.3 Index theory

Deep mathematical work has been done to prove some of the above conjectures in special cases. At present, the most successful approach uses index theory. Consider an operator T on a Hilbert space H which has the property that both its kernel and its cokernel (i.e., the quotient of H by the image of T) are finite-dimensional. Define the Fredholm index of T via the formula

$$\operatorname{index} T = \dim \ker T - \dim \operatorname{coker} T.$$

This quantity is a surprisingly robust invariant; for instance, it is stable under compact perturbations. The Atiyah–Singer index theorem computed the above analytic index of elliptic differential operators on a smooth, compact manifold in terms of cohomology classes of that manifold. We refer the reader to [151] for an extensive overview of the Atiyah–Singer index theorem, as well as to the original papers of Atiyah and Singer [11], [12].

Here we are concerned mainly with non-compact manifolds, and a generalization of the ideas behind the Atiyah–Singer index theorem to that setting requires significant effort. The elliptic operators one has to deal with are not Fredholm in the above sense, and new ideas are necessary to define the analytic index.

In this setting the appropriate language is that of assembly maps from a K-homology group into a K-theory group of an appropriately chosen C*-algebra. Here K-homology can be viewed as a group of generalized elliptic operators, modulo a certain homotopy equivalence. The original Fredholm index is stable under such equivalences.

A more refined equivariant higher index is defined by the Baum–Connes assembly map, whose origins are in the work of Kasparov [141]. The map was constructed by P. Baum and A. Connes [22], [23], [24] and redefined by Baum, Connes and Higson in [25]. The Baum–Connes assembly map is a homomorphism

$$\mu\colon K_*^G(\underline{E}G) \to K_*(C_r^*(G)),$$

where K^G is a version of equivariant K-homology adapted to deal with non-locally compact spaces; $\underline{E}G$ is the universal space for proper actions of G; $C_r^*(G)$ is the reduced group C*-algebra. A detailed definition of the Baum–Connes assembly map is beyond the scope of this survey, and we refer the reader to [169], [232] for an excellent introduction.

Conjecture 8.3.1 (Baum–Connes conjecture). *The Baum–Connes assembly map μ is an isomorphism for every locally compact group G.*

The Baum–Connes conjecture provides an algorithm for computing the equivariant higher index invariants, if true. The conjecture was verified for many classes of groups, most notably including a-T-menable groups [126] and hyperbolic groups [167], [149].

The coarse Baum–Connes map defines a higher index for certain operators on general non-compact manifolds. The ideas begin with the work of Roe, who proved an index theorem on regularly exhaustible manifolds [206]. A regularly exhaustible manifold is one which, roughly speaking, is quasi-isometric to a discrete amenable space, i.e, it can be exhausted by large sets with small boundary. The idea in Roe's work was to use a functional, constructed similarly as an invariant mean on an amenable group, to average some of the index-theoretic information. However, existence of a regular exhaustion is a restrictive condition and many open manifolds do not satisfy it. To overcome this difficulty, Roe introduced coarse homology and cohomology, as discussed in the last chapter. Since the K-homology groups of a manifold are groups of generalized elliptic operators, by applying the coarsening homomorphism to these groups we obtain a homomorphism

$$c\colon K_*(M) \to KX_*(M),$$

where M is an open manifold. The object on the right-hand side is the coarsened K-homology of M and, intuitively, one can think of it as a group of elliptic operators that are stripped of any infinitesimal data and such that only their global, large scale structure remains.

The coarse Baum–Connes assembly is then a geometric version of the Baum–Connes map,

$$\mu^c\colon KX_*(X) \longrightarrow K_*(C^*(X)).$$

Here, $C^*(X)$ denotes the Roe algebra of the space X. This algebra is generated by locally compact operators of finite propagation and for a group it can be viewed as a non-equivariant version of $C_r^*(G)$, with compact coefficients. This point of view was formalized in [246], where a relation between the coarse Baum–Connes map and the usual Baum–Connes map for certain coefficients was exhibited. An important feature of Roe algebra is that is invariant under coarse equivalence. The Roe algebra of a space X encodes the non-commutative large scale geometry of X.

We now formulate the coarse Baum–Connes conjecture. Although false in general (see below), this conjecture provided much inspiration.

Conjecture 8.3.2. *The coarse Baum–Connes assembly map μ^c is an isomorphism for every bounded geometry metric space X.*

While the left-hand side of the coarse Baum–Connes conjecture contains information about the large scale algebraic topology of X, the right-hand inside encodes the non-commutative large scale topology of X. The importance of the coarse Baum–Connes conjecture is in the fact that it provides an algorithm for computing higher index invariants, if true.

The conjecture was formulated in [207], [129], [247] and we refer to these articles for details. When true, the coarse Baum–Connes conjecture has many applications. The most spectacular one is the Novikov conjecture for a finitely generated group G (i.e., for all compact manifolds M with $\pi_1(M) \cong G$). The following result establishes the coarse Baum–Connes conjecture for a large class of metric spaces and groups.

Theorem 8.3.3 ([249]). *The coarse Baum–Connes conjecture is true for every bounded geometry metric space with finite asymptotic dimension.*

Examples of groups to which the above theorem applies are discussed in Chapter 2. Another application of asymptotic dimension to the study of the Novikov conjecture can be found in [136]. As mentioned therein, there are also plenty of examples of groups that have infinite asymptotic dimension. A much stronger theorem was proved in [250].

Theorem 8.3.4. *Let X be a bounded geometry metric space which admits a coarse embedding into a Hilbert space. Then the coarse Baum–Connes conjecture holds for X.*

As discussed in Chapter 5, the class of metric spaces and groups to which this theorem applies is very large. In fact, at present there is only one construction of examples which do not embed coarsely into a Hilbert space. These examples, often called Gromov monsters, were constructed by Gromov using random methods [110], see also [7]. Aspherical manifolds whose fundamental groups contain expanders were constructed in [218]. We refer the reader to [252] for another overview of the subject. We also mention that a groupoid version of the Baum–Connes conjecture and its relation to the coarse Baum–Connes conjecture was studied in [223]. A stronger result on the Novikov conjecture was proved recently in [143], where the embedding is allowed to be into a larger class of Banach spaces, including uniformly convex Banach spaces with an unconditional basis.

The coarse Baum–Connes conjecture is known to be false in general for bounded geometry metric spaces; counterexamples are expander graphs and Gromov monsters [127].

A weaker version of Theorem 8.3.4 holds under more general assumptions.

Theorem 8.3.5. *Let X be a bounded geometry metric space which admits a coarse embedding into a uniformly convex Banach space. Then the coarse Novikov conjecture holds for X, that is, the coarse index map is injective.*

The coarse Novikov conjecture is, in general, not sufficiently strong to conclude the usual Novikov conjecture. However, it still has certain other applications. The zero-in-the-spectrum conjecture asks if for every complete Riemannian manifold, the Laplace–Beltrami operator on L_2-summable differential p-forms contains zero for some $p \geq 1$. The answer is negative in general [86], however the problem is open for uniformly contractible manifolds, e.g., universal covers of closed, aspherical manifolds. The coarse Novikov conjecture implies the zero-in-the-spectrum conjecture. See also [248] for a direct connection between asymptotic dimension and the zero-in-the-spectrum conjecture.

In certain cases the coarse and the classical Novikov conjectures are related. More precisely, consider a residually finite group G and a collection of normal subgroups N_i with trivial intersection. In [98] it is proved that if the coarse Novikov conjecture is true for the coarse disjoint union of $\{G/N_i\}$, then the Baum–Connes map is rationally injective. This last statement is often referred to as the strong Novikov conjecture. The converse implication is also true under additional assumptions.

Significant effort is being put into extending theorems of the above form to more general situations, mainly by including more general Banach spaces as target spaces of coarse embeddings. Recall that by Theorem 5.4.3, every bounded geometry metric space embeds into a reflexive Banach space. On the other hand, there are examples of expander graphs which do not admit a coarse embedding into any uniformly convex Banach space [150], [165]. Such examples are difficult to construct and it is an open question whether there exists a sequence of expanders whose coarse disjoint union allows a coarse embedding into some uniformly convex Banach space (Question 5.6.6).

Nevertheless, the coarse Novikov conjecture for expander graphs is known in several cases. In [194] the maximal version of the coarse Baum–Connes conjecture was proven for certain expanders, arising as coarse disjoint unions of finite quotients of groups. The maximal version of the coarse Baum–Connes conjecture for expanders with girth tending to infinity was proved in [240], [241]. The failure of the coarse Baum–Connes conjecture is also discussed in [127], [240], [241]. Finally, in [57], [116] an operator-theoretic property was studied and applied to prove the coarse Novikov conjecture for many classes of expanders.

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Index